Diagonal Spatial Stiffness Matrices

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Abstract

In this work we study in detail the conditions under which the stiffness matrix of a spatial system can be transformed into block-diagonal and diagonal form. That is the existence of a coordinate frame in which the stiffness matrix takes on these simple forms. The consequences of a block-diagonal or diagonal stiffness matrix for the invariants of the system, principal screws, von Mises' invariants and so forth, are also studied.

Keywords:Stiffness matrix, Spatial compliance, Diagonalisation.

1 Introduction

Stiffness and compliance matrices have been extensively studied by researchers in robotics. There are several reasons for the interest, for instance, passive compliant devices such as the remote-centre-compliance wrist have been used for assembly tasks. Secondly, impedance control schemes have been used in robotics to cope with situations where the robot's end-effector contacts the environment. Finally, it is well known that the links and joints of a robot are to some extent compliant and stiffness matrices can provide a simple first step to modelling these components.

In many cases of practical importance the stiffness matrix of a system turns out to be diagonal. This is true in particularly when the stiffness matrix is selected in the design of a machine or system and is chosen to have a centre of compliance. But it also occurs in natural systems such as beams. In this work we look at the conditions for the stiffness matrix of a system to be diagonal. We also look at some of the implications for the principal screws and force/torque compliance axes of the system. Some results for the values of the von Mises' invariants are also presented. Note that the conditions for a stiffness matrix to have a center of compliance, which is a specialisation of diagonalisable case, was studied in [1]. We approach the problem of diagonalisation in two stages, first we look at the case where the matrix is block diagonalisable. That is, where we can transform the coordinates so that the top-right and bottom-left 3×3 submatrices vanish. Subsequently, we study the case where the top-left and bottom-right submatrices can be simultaneously diagonalised.

As mentioned above, many workers have studied stiffness from the point of view of spatial mechanical systems. Historically Ball [2] was probably the first to look at the problem, using screw theory. He defined

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six principal screws which are the solution to a simple eigenvalue problem. Richard von Mises [3] looked again at the problem. This time as an application of his "motor calculus". Von Mises described a system of 15 invariants of the stiffness matrix. These invariants are polynomials in the entries of the stiffness matrix, whose values are independent of the coordinate frame in which the matrix is expressed. Josip Lončarić [4], a student of Brockett, described a normal form for the stiffness matrix. He showed that almost all spatial stiffness matrices can be transformed into a fairly simple shape by applying a suitable rigid transformation. More recently, Patterson and Lipkin [5, 6] have introduced the idea of force-compliant and torque-compliant axes. These were derived from earlier ideas studied by Dimentberg [7]. The problem of synthesizing a given stiffness matrix using a small number of physical components such as springs has been studied by Huang and Schimmels [8] and Roberts [9]. This paper builds on the work in [1] where many of the above ideas were reviewed.

We begin by introducing some notation.

2 Notation

Consider a rigid body in a potential field ϕ. This potential might be due to gravity, electro-magnetic effects or any physical cause, but perhaps the main motivation here comes from the case where the body is suspended by a compliant mechanism of beams or springs.

The potential function is a function on the group of rigid body motions $SE(3)$, in general as the body translates or rotates the potential changes, $\phi : SE(3) \longrightarrow \mathbb{R}$.

The generalised force is given as the gradient of the potential in the usual way,

$$\mathcal{W} = -d\Phi$$

where d is the exterior derivative operator. The generalised forces are cotangent vectors or one-forms. In fact, since the configuration manifold of a rigid body is a Lie group, we can think of the forces as elements of the dual to the Lie algebra. These vectors are also called wrenches. We can partition the wrenches into force and torque vectors, $\mathcal{W}^T = (\boldsymbol{\tau}^T,\, \mathbf{F}^T)$.

At an equilibrium configuration, where $\mathcal{W} = \mathbf{0}$, the Hessian of the potential energy defines a symmetric tensor. This is the stiffness matrix, K. In partitioned form stiffness matrix has the form:

$$K = \begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix}$$

In particular cases where the potential is given explicitly the stiffness matrix can be derived directly from the potential by differentiating, see [10] for example.

The stiffness matrix maps twists to wrenches. A twist here is a small displacement, an element of the Lie algebra of $SE(3)$. In a Cartesian coordinate frame we can partition twists into an angular displacement $\boldsymbol{\omega}$, and a linear displacement $\mathbf{v}$. So the full six-dimensional twist is given by $\mathbf{s}^T = (\boldsymbol{\omega}^T,\, \mathbf{v}^T)$. These vectors will also be called screws here. Thus to produce a small displacement $\mathbf{s}^T = (\boldsymbol{\omega}^T,\, \mathbf{v}^T)$, the wrench we must apply is given by,

$$\begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix} = K \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$$

Alternatively, we can interpret this formula as giving us the displacement produced by a specified wrench.

Hence, if the motion of the body is given by a twist $\mathbf{s}_2$ when a twist $\mathbf{s}_1$ is applied, the work done is given by the bilinear form,

$$\text{Work} = \mathbf{s}_2^T K \mathbf{s}_1$$

The stiffness matrix has much in common with the inertia matrix for a rigid body. Like the inertia matrix it is a 6×6 symmetric matrix. However, whereas the inertia matrix is fairly tightly constrained by mechanics, the stiffness matrix can be any 6×6 symmetric matrix, depending on the potential.

A rigid change of coordinates transforms the stiffness matrix according to,

$$K' = H^T K H$$

where H is a rigid transformation with the partitioned form,

$$H = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}$$

here R is a 3×3 rotation matrix and T is an anti-symmetric matrix representing the translation, $T\mathbf{x} = \mathbf{t} \times \mathbf{x}$ for any 3-vector $\mathbf{x}$. This can be derived from the transformation properties of the twist and wrench, the transformation of the stiffness matrix has to be chosen so that the work given in the section above is a scalar, that is independent of coordinate transformations. The transformation relation for the stiffness matrix is, of course, exactly the same as that for the inertia matrix of a rigid body.

In terms of the 3×3 submatrices the transformation relations can be written as,

$$\begin{aligned} \Xi' &= R^T \Xi R + R^T \Gamma T R - R^T T \Gamma^T R - R^T T \Upsilon T R \\ \Gamma' &= R^T \Gamma R - R^T T \Upsilon R \\ \Upsilon' &= R^T \Upsilon R \end{aligned}$$

Lončarić [4] showed that it was almost always possible to choose coordinates so that the submatrices Ξ and Υ are symmetric and the matrix Γ is diagonal. See also [1].

We will also use the following partitioned matrices,

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \quad Q_\infty = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

where I_3 is the 3×3 identity matrix. These matrices can also be thought of as mappings from twists to wrenches, but under proper rigid coordinate changes they are invariant,

$$H^T Q_0 H = Q_0, \quad H^T Q_\infty H = Q_\infty.$$

Moreover any other matrix with this property has the form, $\lambda Q_0 + \mu Q_\infty$ for some constants $\lambda,\ \mu$, see [11] for further details.

3 Block Diagonal Form

Here we study the conditions for a stiffness matrix to be equivalent to a block diagonal matrix, that is one which can be transformed to a coordinate frame where $\Gamma = 0$. Clearly a rotation cannot effect such a change so we only need to look at translations. We seek the conditions for which a translation T exists that satisfies the equation,

$$\Gamma - T\Upsilon = 0.$$

The simplest characterisation of this form is as follows. A stiffness matrix is block-diagonalisable if and only if it has an inversion symmetry.

An inversion about the origin acts on points by reversing the signs of their components, that is the inversion is represented by the matrix $-I_3$, extending this to screws we get the 6×6 matrix representation,

$$F_o = \begin{pmatrix} -I_3 & 0 \\ 0 & I_3 \end{pmatrix}.$$

The inversion acts on a stiffness matrix by a congruence, so a stiffness matrix invariant under this inversion satisfies,

$$K = F_o^T K F_o.$$

Now suppose we had a stiffness matrix with an inversion symmetry with respect to a point with position vector $\mathbf{t}$. We can derive the matrix representing this inversion by translating the inversion centre back to the origin, performing the inversion through the origin, as above and then translating back to the inversion centre:

$$F_t = \begin{pmatrix} I_3 & 0 \\ T & I_3 \end{pmatrix} \begin{pmatrix} -I_3 & 0 \\ 0 & I_3 \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ -T & I_3 \end{pmatrix} = \begin{pmatrix} -I_3 & 0 \\ -2T & I_3 \end{pmatrix}.$$

Now the equation,

$$K = F_t^T K F_t$$

can be expanded in partitioned form to give, $\Gamma = -\Gamma + 2T\Upsilon$ in the top right-hand corner. This has the solution $\Gamma = T\Upsilon$ showing that Γ will disappear if we transform the origin to the centre of the inversion. The other equations, from the other corners of the partitioned matrix, are then automatically satisfied.

Conversely, assume that the stiffness matrix K is block-diagonaliseable. When it is transformed to block-diagonal form it will be clearly invariant under inversion through the origin. Hence, in the original coordinates it must be invariant with respect to an inversion in the same point, but expresses in the old coordinates.

Hence, a stiffness matrix is block diagonaliseable if and only if it has an inversion symmetry.

Although this characterisation is geometrically elegant it is not very easy to use computationally. So we look for another characterisation more closely related to the elements of K.

If K is block-diagonalisable then there will be a translation $\mathbf{t}$, or T as an anti-symmetric matrix, which satisfies,

$$\Gamma - T\Upsilon = 0.$$

The above matrix equation represents 9 linear equations for the 3 components of $\mathbf{t}$. Transposing the above equation gives,

$$\Gamma^T + \Upsilon T = 0$$

since Υ is symmetric and T is anti-symmetric. Pre-multiplying the first of the these equations by Υ and post-multiplying the second by Υ and then adding we eliminate T to get,

$$\Upsilon\Gamma + \Gamma^T\Upsilon = 0.$$

The left hand side of this equation is a 3×3 symmetric matrix and hence the equation represents 6 conditions on the entries of K. These are clearly necessary conditions for a solution $\mathbf{t}$, to exist. In the case that $\det(\Upsilon) \neq 0$ it can also be seen that the conditions are sufficient, since they imply that $T = \Gamma\Upsilon^{-1}$ is anti-symmetric.

Notice also that, when $\det(\Upsilon) \neq 0$, the condition $\Upsilon\Gamma + \Gamma^T\Upsilon = 0$ implies that $\det(\Gamma) = 0$. Finally notice that the matrix $\Upsilon\Gamma + \Gamma^T\Upsilon$ is invariant with respect to translations but transforms according to the usual congruence under rotations. In old terminology such a matrix would be called a covariant of the system.

4 Eigenscrews

In this section we relate the results found above to previous work on stiffness matrices.

First we look at the principal screws or eigenscrews of the block-diagonal stiffness matrix. These screws were defined by Ball [2] as the screws which satisfy the relation,

$$K\mathbf{s} = \lambda Q_0 \mathbf{s}$$

where λ is the eigenstiffness of the screw. Now suppose that K has an inversion symmetry F_t, for every principal screw $\mathbf{s}$ there is another principal screw $F_t\mathbf{s}$.

$$KF_t\mathbf{s} = F_t^T K\mathbf{s} = \lambda F_t^T Q_0 \mathbf{s} = -\lambda Q_0 F_t \mathbf{s}$$

since $F_t^T Q_0 F_t = -Q_0$ and $F_t F_t = I_6$ the 6×6 identity matrix. From this it is clear that $KF_t\mathbf{s} = -\lambda Q_0 F_t\mathbf{s}$, that is $F_t\mathbf{s}$ is a principal screw with eigenstiffness $-\lambda$.

In general the pitch of a screw $\mathbf{s}$, is given by the expression, $\pi = \mathbf{s}^T Q_0 \mathbf{s}/2\mathbf{s}^T Q_\infty \mathbf{s}$. From this is it easy to see that the principal screw $F_t\mathbf{s}$ has the opposite pitch to $\mathbf{s}$. Notice that $F_t^T Q_\infty F_t = Q_\infty$.

Furthermore, the principal screws $\mathbf{s}$ and $F_t\mathbf{s}$ have parallel axes. To see this, write $\mathbf{s}^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$ then,

$$F_t\mathbf{s} = \begin{pmatrix} -I_3 & 0 \\ -2T & I_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\omega} \\ \mathbf{v} - 2\mathbf{t} \times \boldsymbol{\omega} \end{pmatrix}$$

which is clearly parallel to the original $\mathbf{s}$.

Next we turn to the compliant axes of Patterson and Lipkin [5, 6]. These are defined by eigenvector problems in a similar way to the principal screws seen above. Force-compliant axes satisfy,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \eta \begin{pmatrix} \mathbf{c} \times \mathbf{v} \\ \mathbf{v} \end{pmatrix}$$

When the stiffness matrix is block-diagonal we get 3-solutions, corresponding to the 3 eigenvectors of Υ. The force-compliant axes all pass through the origin, or more generally, through the inversion center. Since the directions of the axes are eigenvectors of a symmetric matrix, they will be mutually orthogonal, if Υ has no repeated eigenvalues.

The torque-compliant axes satisfy,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{r} \times \boldsymbol{\omega} \end{pmatrix} = \mu \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{0} \end{pmatrix}$$

Again it is simple to see that there will be 3 solutions for a block-diagonal stiffness matrix, this time corresponding to the eigenvectors of Ξ. Again the axes will pass through inversion centre and will be mutually orthogonal.

Patterson and Lipkin also defined compliant axes which are both force-compliant and torque-compliant axes. Clearly, block-diagonal stiffness matrices have no compliant axes unless Ξ and Υ have a common eigenvector.

5 Diagonal Stiffness Matrices

As in the case of block-diagonalisable stiffness matrices there is an elegant geometrical characterisation of the diagonalisable matrices.

If the stiffness matrix has three orthogonal plane reflection symmetries then it is fully diagonalisable. To see this assume that the reflection planes are the x, y and z-planes in some coordinate system. The 6×6 matrices representing these reflections are,

$$F_i = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$F_j = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$F_k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that $F_i F_j F_k = F_o$ so a stiffness matrix with these three reflection symmetries must also have an inversion symmetry. Hence, as we might expect, if a stiffness matrix is diagonalisable it is also block diagonalisable. To prove the result we look at the effect of a reflection on the stiffness matrix. Any matrix, invariant with respect to a reflection in yz-plane will satisfy,

$$K = F_i^T K F_i.$$

Substituting for F_i and using general elements for K we see that K must have the form,

$$K = \begin{pmatrix} x_{11} & 0 & 0 & 0 & g_{12} & g_{13} \\ 0 & x_{22} & x_{23} & g_{21} & 0 & 0 \\ 0 & x_{23} & x_{33} & g_{31} & 0 & 0 \\ 0 & g_{21} & g_{31} & u_{11} & 0 & 0 \\ g_{12} & 0 & 0 & 0 & u_{22} & u_{23} \\ g_{13} & 0 & 0 & 0 & u_{23} & u_{33} \end{pmatrix}$$

Repeating this for the reflections F_j and F_k shows that the stiffness matrix has to be diagonal.

Again we seek a computationally simpler characterisation of these matrices. Assuming that we have a block diagonalisable stiffness matrix, what extra requirements must it satisfy in order that it be completely diagonalisable? We must effect the diagonalisation with a rotation, since a translation would disturb the elements of the off-diagonal block. Hence, the problem reduces to a familiar one: Under what circumstances are a pair of symmetric matrices Ξ and Υ, simultaneously diagonalisable? The necessary and sufficient condition is well known, see Cohn [12, sect. 11.5] for example, we must have that Ξ and Υ commute,

$$\Xi\Upsilon - \Upsilon\Xi = 0.$$

This relation only holds in a coordinate frame where the stiffness matrix has block-diagonal form. Let us write this relation in terms of the elements of a general stiffness matrix. We will assume that the matrix is block diagonalisable so let T be the translation that block diagonalises K. That is, $\Gamma = T\Upsilon$. Translating to block-diagonal form and applying the commutation relation above gives,

$$\begin{aligned} 0 &= (\Xi + \Gamma T - T\Gamma^T - T\Upsilon T)\Upsilon - \Upsilon(\Xi + \Gamma T - T\Gamma^T - T\Upsilon T) \\ &= (\Xi\Upsilon - \Upsilon\Xi) + (\Gamma - T\Upsilon)T\Upsilon + \Upsilon T(\Gamma^T + \Upsilon T) - \Upsilon\Gamma T - T\Gamma^T\Upsilon \\ &= \Xi\Upsilon - \Upsilon\Xi + \Gamma^2 - (\Gamma^T)^2 \end{aligned}$$

In summary, for a stiffness matrix to be diagonalisable we must have:

$$\begin{aligned} \Upsilon\Gamma + \Gamma^T\Upsilon &= 0 \\ \Xi\Upsilon - \Upsilon\Xi + \Gamma^2 - (\Gamma^T)^2 &= 0 \end{aligned}$$

when $\det(\Upsilon) \neq 0$.

These relations can be expressed rather neatly in terms of the matrix $K^{\hat{2}} = KQ_0K$. This matrix is a sort of square of the original stiffness matrix K. In terms of the partitioned form of the stiffness matrix we have,

$$K^{\hat{2}} = \begin{pmatrix} \Xi\Gamma^T + \Gamma\Xi & \Xi\Upsilon + \Gamma^2 \\ \Upsilon\Xi + (\Gamma^T)^2 & \Upsilon\Gamma + \Gamma^T\Upsilon \end{pmatrix}.$$

The bottom right block of this matrix will vanish if K is block-diagonalisable and the two off-diagonal blocks of $K^{\hat{2}}$ will be symmetric if K is fully diagonalisable. Moreover, when K is block-diagonalisable it is possible to find a translation which makes the top left block vanish, of course it is the same translation which makes Γ' disappear in the K.

The principal screws of the diagonal stiffness matrix are straightforward to find. The three pairs of principal screws we found in the block-diagonal case now become three orthogonal axes meeting at a point. With diagonal matrix entries x_1, x_2, x_3, u_1, u_2 and u_3 the axes of the principal screws are directed along the coordinate axes with pitches,

$$\pm\sqrt{\frac{x_1}{u_1}}, \quad \pm\sqrt{\frac{x_2}{u_2}}, \quad \pm\sqrt{\frac{x_3}{u_3}},$$

and eigenstiffnesses,

$$\pm\sqrt{x_1u_1}, \quad \pm\sqrt{x_2u_2}, \quad \pm\sqrt{x_3u_3}.$$

Patterson and Lipkin [6] show that their compliant axes correspond to pairs of principal screws sharing a common axis but with opposite pitches and eigenstiffnesses. Hence, it is simple to see that diagonalisable stiffness matrices have 3 orthogonal compliant axes. This means that Patterson and Lipkin's class 3a is precisely the class of diagonalisable stiffness matrices.

6 Von Mises Invariants

In 1924 Ricard von Mises found the full set of invariants for spatial stiffness matrices. That is, he found a set of 15 polynomials in the matrix entries, invariant with respect to rigid coordinate changes. All other invariants of the stiffness matrix are functions of this basic set. Contemporary workers in the field of stiffness and compliance have not made much use of this ground breaking work. In this section we attempt to redress this to some extent. After briefly describing the invariants we look at the problem of diagonalisability once again, this time in terms of the von Mises invariants.

Consider the degree 6 polynomial in λ and μ defined by,

$$\det\big(K - \lambda Q_0 - \mu Q_\infty\big) = 0.$$

Here K is the stiffness matrix as usual, Q_0 and Q_∞ are as defined in section 2 above.

The transformation properties of these matrices mean we can write,

$$\det\big(H^T K H - \lambda Q_0 - \mu Q_\infty\big) = \det(H^T)\det\big(K - \lambda Q_0 - \mu Q_\infty\big)\det(H) = \det\big(K - \lambda Q_0 - \mu Q_\infty\big).$$

So the polynomial is invariant with respect to coordinate changes. Now, since λ and μ are arbitrary the coefficient of each monomial in λ and μ are invariant with respect to rigid motions.

$$\begin{aligned}\det\big(K - \lambda Q_0 - \mu Q_\infty\big) \;=\;& -\lambda^6 + b_1\lambda^5 - (b_2 + a_1\mu)\lambda^4 + (b_3 + b_3'\mu)\lambda^3 - (b_4 - b_4'\mu + a_2\mu^2)\lambda^2 + \\ & (b_5 - b_5'\mu + b_5''\mu^2)\lambda + b_6 - b_6'\mu + b_6''\mu^2 - a_3\mu^3 \qquad (\star)\end{aligned}$$

The 15 coefficient a_i, b_i and so forth, are the von Mises' invariants.

It is possible to find expressions for these invariants in terms of sums of sub-determinants of K. For example, $b_6 = \det(K)$ and $a_3 = \det(\Upsilon)$. A complete list of these invariants is given in [1].

Our first result is that, if the stiffness matrix is symmetric under a reflection, that is an element of $E(3)$ not in $SE(3)$, then the six von Mises' invariants $b_1 = b_3 = b_5 = b_3' = b_5' = b_5'' = 0$ vanish.

We may represent such a reflection by a partitioned matrix,

$$F = \begin{pmatrix} M & 0 \\ TM & -M \end{pmatrix}$$

where $M \in O(3)$ is an orthogonal matrix with $\det(M) = -1$, and T is an anti-symmetric translation matrix as before. Straightforward computations reveal that,

$$F^T Q_0 F = -Q_0 \qquad \text{and} \qquad F^T Q_\infty F = Q_\infty$$

Transforming equation ($\star$) above we have,

$$\begin{aligned}\det(K - \lambda Q_0 - \mu Q_\infty) &= \det(F^T(K - \lambda Q_0 - \mu Q_\infty)F) \\ &= \det(K + \lambda Q_0 - \mu Q_\infty)\end{aligned}$$

Hence all the odd coefficients of λ must vanish, including those involving μ.

These six conditions are not enough to guarantee that a stiffness matrix is block-diagonalisable. Certainly if a stiffness matrix has an inversion symmetry then these six invariants will vanish. However, a stiffness matrix which has a single plane reflection symmetry F_i say, as in section 5 above, will also have $b_1 = b_3 = b_5 = b_3' = b_5' = b_5'' = 0$ but cannot necessarily be transformed to block-diagonal form.

The characteristic equation of a diagonal stiffness matrix factorises,

$$\det\begin{pmatrix} x_1-\mu & 0 & 0 & -\lambda & 0 & 0 \\ 0 & x_2-\mu & 0 & 0 & -\lambda & 0 \\ 0 & 0 & x_3-\mu & 0 & 0 & -\lambda \\ -\lambda & 0 & 0 & u_1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & u_2 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & u_3 \end{pmatrix} = -(\lambda^2 + \mu u_1 - x_1u_1)(\lambda^2 + \mu u_2 - x_2u_2)(\lambda^2 + \mu u_3 - x_3u_3)$$

where the x_i and u_i are the diagonal entries of the stiffness matrix. Expanding this and comparing terms with equation ($\star$) above, the von Mises' invariants can be read off,

$$\begin{aligned}
a_1 &= u_1 + u_2 + u_3 \\
a_2 &= u_1u_2 + u_3u_1 + u_2u_3 \\
a_3 &= u_1u_2u_3 \\
b_2 &= -(x_1u_1 + x_2u_2 + x_3u_3) \\
b_4 &= x_1x_2u_2u_3 + x_3x_1u_3u_1 + x_2x_3u_2u_3 \\
b_6 &= -(x_1x_2x_3)u_1u_2u_3 \\
b_4' &= x_1(u_1u_2 + u_3u_1) + x_2(u_1u_2 + u_2u_3) + x_3(u_3u_1 + u_2u_3) \\
b_6' &= -(x_1x_2 + x_2x_3 + x_3x_1)u_1u_2u_3 \\
b_6'' &= -(x_1 + x_2 + x_3)u_1u_2u_3
\end{aligned}$$

with $b_1 = b_3 = b_5 = b_3' = b_5' = b_5'' = 0$, of course.

Notice that in the case where $\Xi = \alpha I_3$ and $\Upsilon = \beta I_3$, the classical case where a centre of compliance exist, then the characteristic equation is a perfect cube.

It is not clear whether the fact that the characteristic equation factorises ensures that the stiffness matrix is diagonalisable.

Under a rigid transformation $K^{\hat{2}}$ behaves in exactly the same way as a stiffness matrix,

$$(K^{\hat{2}})' = H^T K H Q_0 H^T K H = H^T K^{\hat{2}} H,$$

since $HQ_0H^T = Q_0$. This means that $K^{\hat{2}}$ will have von Mises invariants, let us call them $\hat{a}_1, \ldots, \hat{b}_1, \ldots$ and so forth. In terms of these invariants, the conditions for block diagonalisability and full-diagonalisability are not too difficult to write down. For example, the condition for block-diagonalisability is that the bottom right hand block is zero, this is equivalent to requiring that $\hat{a}_1 = \hat{a}_2 = \hat{a}_3 = 0$.

According to von Mises the 15 invariants $a_1, \ldots b_2, \ldots$ form a basis for all possible invariants of the stiffness matrix. So it should be possible to write the invariants of $K^{\hat{2}}$ in terms of the invariants of K. That is, $\hat{a}_1, \ldots, \hat{b}_1, \ldots$ should be polynomials in the "un-hatted" invariants $a_1, \ldots b_2, \ldots$. Some of these relations are straightforward, for example the $\hat{b}_i$s. Notice that the matrix $K^{\hat{2}}$ has the same principal screws as K, but the corresponding eigenstiffnesses are the squares of the original eigenstiffnesses, suppose $\mathbf{s}$ is a principal screw for K with eigenstiffness λ,

$$K^{\hat{2}}\mathbf{s} = KQ_0K\mathbf{s} = \lambda K Q_0^2 \mathbf{s} = \lambda^2 Q_0 \mathbf{s}$$

since $Q_0^2 = I_6$. Since the b_i invariants are symmetric polynomials in these eigenstiffnesses it is relatively simple to find, $\hat{b}_1 = b_1^2 - 2b_2$, $\hat{b}_2 = b_2^2 - b_1b_3, \ldots, \hat{b}_6 = b_6^2$.

Also we have that $\hat{a}_1 = \mathrm{Tr}(\Upsilon\Gamma + \Gamma^T\Upsilon) = 2\,\mathrm{Tr}(\Upsilon\Gamma) = a_3b_1 - b_3'$, see [1]. However, it seem to be rather more difficult to find expressions for the other hatted invariants in terms of the original ones.

7 Conclusions

In the above we have given several different characterisations of block-diagonalisable and diagonalisable stiffness matrices. The characterisation in terms of the existence of particular kinds of symmetries seems to be the most elegant. Whereas the conditions given in terms of relations involving the 3×3 blocks of the stiffness matrix appear to the the most practical in terms of computation. We believe that both these results are novel.

We have been able to find out quite a lot about about the von Mises invariants in the cases under consideration. Enough we hope, to suggest that these are important concepts, worthy of further study.

More generally, we should expect that possible simplifications of the stiffness matrix will be determined by symmetry conditions. It should also be possible to express the conditions for simplification in terms of invariants or covariants of the matrix. The von Mises' invariants used above, may not be the most convenient to use. Hence, a detailed study of the invariants of the striffness matrix would seem to be called for.

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