Structure of the Spatial Stiffness Matrix

J.M. Selig and X. Ding*
School of Computing, Information Systems & Mathematics
South Bank University
London SE1 0AA, U.K.

Abstract

In this article we review several approaches to study the spatial stiffness matrix. These include the principal screws of Ball, Lončarić's normal form and the compliance axes of Patterson and Lipkin. In addition we reintroduce the invariants found by von Mises in the 1920s. We attempt to describe the work in a common mathematical framework and to make connections between the different viewpoints.

1 Introduction

This article collects together some well known and some less well known results concerning the structure of the spatial stiffness matrix. The idea is to present all of this work in a common framework and notation so that it becomes clear how the different views relate to each other.

Historically Ball [1] was probably the first to look at the problem, using screw theory. He defined six principal screw which are the solution to a simple eigenvalue problem.

In the 1920s Richard von Mises [2], a student of Study, looked again at the problem. This time as an application of his "motor calculus". Von Mises was able to describe a system of 15 invariants of the stiffness matrix. These invariants completely determine the stiffness matrix up to an overall rigid transformation. Sadly the English-speaking world has neglected this work.

It was not until the late 1980s that significant work began again in this area, Josip Lončarić [3], a student of Brockett, described a normal form for the stiffness matrix. That is, almost all spatial stiffness matrices can be transformed into a fairly simple shape by applying a suitable rigid transformation.

Most recently, Patterson and Lipkin [4, 5] have introduced the idea of force-compliant and torque-compliant axes. These were derived from ideas studied by Dimentberg citeDimentberg in the 1960s. Further, Patterson and Lipkin, made important contributions to the case where the stiffness matrix determines a stable equilibrium configuration of the rigid body.

These ideas are relevant to the study of compliant grasps. In this context Lin, Burdick and Rimon [6] attempted to define a quality measure for a compliant grasp. This was based on the stiffness matrix of the system and some simple geometric properties of the grasped object. This work independently found similar results to that of Patterson and Lipkin. Another approach in this area was given by Bruyninckx, Demy and Kumar [7], here the stiffness matrix of the system was combined with an arbitrary inertia matrix to produce invariants. These ideas seem to be most appropriate when the inertia matrix can be identified with the inertia matrix of the grasped object.

*permanent address: Robotics Research Institute, Beijing University of Aero.& Astro., P. R. China

The general setting is as follows. Consider a rigid body subject to a potential field. This potential may have any cause, electrical, magnetic, gravitational and so forth, but perhaps the main motivation is the case where the potential is due to a number of springs. These might even be the compliant fingers of a robot gripper.

The potential function is a function on the group of rigid body motions $SE(3)$, as the body translates or rotates the potential changes in general. The force and torque can be combined into a single 6-dimensional vector called a wrench. The wrench on the body due to the potential is given by the gradient of the potential as usual,

$$\mathcal{W} = -d\Phi$$

This implies that wrenches are cotangent vectors. In fact, since the configuration manifold of a rigid body is a Lie group, we can think of wrenches as elements of the dual to the Lie algebra.

At an equilibrium configuration, where $\mathcal{W} = \mathbf{0}$, the Hessian of the potential energy defines a symmetric tensor. This is the stiffness matrix, K.

In partitioned form the stiffness matrix has the form:

$$K = \begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix}$$

In particular cases where the potential is given explicitly the stiffness matrix can be derived directly from the potential by differentiating, see [8] for example.

The stiffness matrix maps twists to wrenches. A twist here is a small displacement, an element of the Lie algebra of $SE(3)$. In a Cartesian coordinate frame we can partition twists into an angular displacement $\boldsymbol{\omega}$, and a linear displacement $\mathbf{v}$. So the full six-dimensional twist is given by $\mathbf{s}^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$. These vectors will also be called screws here. The wrench is also a six-dimensional vector, but this time an element of the dual to the Lie algebra. In the same coordinate system, we can partition the wrenches into force and torque vectors, $\mathcal{W}^T = (\boldsymbol{\tau}^T, \mathbf{F}^T)$. Thus to produce a small displacement $\mathbf{s}^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$, the wrench we must apply is given by,

$$\begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix} = K \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$$

Alternatively, we can interpret this formula as giving us the displacement produced by a specified wrench.

Hence, if the motion of the body is given by a twist $\mathbf{s}_2$ when a twist $\mathbf{s}_1$ is applied, the work done is given by the bilinear form,

$$\text{Work} = \mathbf{s}_2^T K \mathbf{s}_1$$

The stiffness matrix has much in common with the inertia matrix for a rigid body. Like the inertia matrix it is a 6×6 symmetric matrix. However, whereas the inertia matrix is fairly tightly constrained by mechanics, the stiffness matrix can be any 6×6 symmetric matrix, depending on the potential.

2 Lončarić Normal Form

A rigid change of coordinates transforms the stiffness matrix according to,

$$K' = H^T K H$$

where H is a rigid transformation with the partitioned form,

$$H = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}$$

here R is a 3×3 rotation matrix and T is an anti-symmetric matrix representing the translation, $T\mathbf{x} = \mathbf{t} \times \mathbf{x}$ for any 3-vector $\mathbf{x}$. This can be derived from the transformation properties of the twist and wrench, the transformation of the stiffness matrix has to be chosen so that the work given in the section above is a scalar, that is independent of coordinate transformations. The transformation relation for the stiffness matrix is, of course, exactly the same as that for the inertia matrix of a rigid body.

In terms of the 3×3 submatrices the transformation relations can be written as,

$$\begin{aligned}\Xi' &= R^T \Xi R + R^T \Gamma T R - R^T T \Gamma^T R - R^T T \Upsilon T R \\ \Gamma' &= R^T \Gamma R - R^T T \Upsilon R \\ \Upsilon' &= R^T \Upsilon R\end{aligned}$$

In [3], Lončarić showed that it was almost always possible to choose coordinates so that the submatrices Ξ and Υ are symmetric and the matrix Γ is diagonal.

To see this, Lončarić split Γ into its symmetric and anti-symmetric parts, $\Gamma_s = \frac{1}{2}(\Gamma + \Gamma^T)$ and $\Gamma_a = \frac{1}{2}(\Gamma - \Gamma^T)$. The anti-symmetric part Γ_a, can be written as a 3-dimensional vector and in this representation it has the transformation properties:

$$2\boldsymbol{\gamma}'_a = R(2\boldsymbol{\gamma}_a - \mathrm{Tr}(\Upsilon) I_3 \mathbf{t} + \Upsilon \mathbf{t})$$

where $\mathbf{t}$ is the vector corresponding to the anti-symmetric matrix T, that is the translation vector. Now we can set $\mathbf{t}$ to be the solution to the system of linear equations,

$$(\mathrm{Tr}(\Upsilon) I_3 - \Upsilon)\mathbf{t} = 2\boldsymbol{\gamma}_a$$

when this is possible the anti-symmetric matrix Γ'_a will be zero and hence Γ' itself will be symmetric. Now we can choose the rotation matrix R in the usual way to diagonalise Γ'. This procedure will only fail when $(\mathrm{Tr}(\Upsilon) I_3 - \Upsilon)$ is singular, that is when $\mathrm{Tr}(\Upsilon)$ is and eigenvalue of Υ. In all other cases we will be able to produce a stiffness matrix with Ξ and Υ symmetric and Γ diagonal.

This is not the only possible normal form, notice that after we have chosen T to make Γ symmetric as above, we could have chosen R to diagonalise Υ or Ξ.

Notice that, unlike the case of inertia matrices, if the submatrix Γ is antisymmetric in some coordinate frame that doesn't necessarily mean that we can transform it to be zero. This is essentially because the combination $T\Upsilon$ that occurs in the transformation law for Γ is not necessarily antisymmetric.

A related question is, under what circumstances can we transform Γ so that it becomes antisymmetric? To do this we would have to solve:

$$(\Gamma - T\Upsilon) + (\Gamma^T + \Upsilon T) = 0$$

for the elements of the antisymmetric matrix T. Now if we rotate the frame so that Υ is diagonal which is always possible, then the equations we have to solve become,

$$\begin{aligned}\gamma_{11} &= 0 & \qquad (u_{11} - u_{22})t_z &= \gamma_{12} + \gamma_{21} \\ \gamma_{22} &= 0 & \qquad (u_{33} - u_{11})t_y &= \gamma_{13} + \gamma_{31} \\ \gamma_{33} &= 0 & \qquad (u_{22} - u_{33})t_x &= \gamma_{23} + \gamma_{32}\end{aligned}$$

where the γ_{ij}s are entries of Γ, u_{ij} are the entries in Υ and t_α are the components of the translation T. The first three equations here give us conditions for the transformation to exist while the final three give the components of the translation.

3 Von Mises' Invariants

Rather than describe a normal form for stiffness matrices, we could look for invariant functions of the matrix entries. That is, functions whose values do not depend on the coordinate frame in which the stiffness matrix is expressed. For example the determinant of K is an invariant since $\det(H) = 1$. In [2] von Mises found 15 invariants and showed that this gave an exhaustive list for the basis of all possible invariants on the space of stiffness matrices.

It is well known that the two matrices,

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \qquad \text{and} \qquad Q_\infty = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

are invariant under the action of the group of rigid body motions. That is, $H^T Q_0 H = Q_0$ and $H^T Q_\infty H = Q_\infty$. Moreover any matrix with this property has the form, $\lambda Q_0 + \mu Q_\infty$ for some constants $\lambda,\ \mu$, see [9] for example. So we have that,

$$\det\left(H^T K H - \lambda Q_0 - \mu Q_\infty\right) = \det(H^T)\det\left(K - \lambda Q_0 - \mu Q_\infty\right)\det(H) = \det\left(K - \lambda Q_0 - \mu Q_\infty\right).$$

This gives an invariant polynomial on the variables λ and μ. Now, since λ and μ are arbitrary the coefficient of each monomial in λ and μ are invariant with respect to rigid motions.

$$\det\left(K - \lambda Q_0 - \mu Q_\infty\right) = -\lambda^6 + b_1\lambda^5 - (b_2 + a_1\mu)\lambda^4 + (b_3 + b_3'\mu)\lambda^3 - (b_4 - b_4'\mu + a_2\mu^2)\lambda^2 + (b_5 - b_5'\mu + b_5''\mu^2)\lambda + b_6 - b_6'\mu + b_6''\mu^2 - a_3\mu^3$$

The 15 coefficient $a_i,\ b_i$ and so forth, are the von Mises' invariants.

It is possible to find expressions for these invariants in terms of sums of sub-determinants of K. This can be done by taking derivatives of the expression above and then setting $\lambda = \mu = 0$. For example, if we simply set $\lambda = \mu = 0$ without any differentiation we have that,

$$b_6 = \det(K)$$

Similarly, if we differentiate with respect to μ three times and then set $\mu = \lambda = 0$ we get,

$$\frac{1}{3!}\frac{\partial^3}{\partial\mu^3}\det\left(K - \lambda Q_0 - \mu Q_\infty\right)\Big|_{\mu=\lambda=0} = \det\begin{pmatrix} -I_3 & \Gamma \\ 0 & \Upsilon \end{pmatrix}$$

Hence we have that, $a_3 = \det(\Upsilon)$. A complete list of these invariants is given in the appendix.

In the following three subsections we look at examples of other invariants which occur naturally in the study of the stiffness matrix and show how these can be written in terms of the von Mises' invariants.

3.1 Lončarić's Invariant

In the derivation of the Lončarić normal form above we encountered the quantity, $\det(\mathrm{Tr}(\Upsilon)I_3 - \Upsilon)$. This quantity had to be non-zero in order that the Lončarić normal form could be found. It is not difficult to see that this quantity is invariant with respect to rigid coordinate transformations, so we will call it the Lončarić invariant and write it as $L = \det(\mathrm{Tr}(\Upsilon)I_3 - \Upsilon)$. The question we seek to answer here is: What is L in terms of the von Mises invariants?

Choose a normal form for the stiffness matrix where the submatrix Υ is diagonal,

$$\Upsilon = \begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{pmatrix}$$

Alternatively, we could think of the matrix entries u_{11}, u_{22} and u_{33} as the eigenvalues of Υ. Now, for this matrix we can simply calculate,

$$L = u_{11}^2(u_{22} + u_{33}) + u_{22}^2(u_{11} + u_{33}) + u_{33}^2(u_{11} + u_{33}) + 2u_{11}u_{22}u_{33}$$

Also a simple computation confirms that, in terms of the von Mises' invariants we have,

$$L = a_1a_2 - a_3$$

Although this relation was derived in a particular coordinate system it relates invariants and hence must be valid in all coordinate frames.

3.2 Force-Compliance Invariants

In a later section of this work we will encounter a condition for force-compliant axes to exist. This is given by the vanishing of the quantity, $\mathbf{v}^T\Gamma\mathbf{v}$, where $\mathbf{v}$ is a unit length eigenvector of Υ. Hence we have three quantities,

$$f_1 = \mathbf{v}_1^T\Gamma\mathbf{v}_1, \quad f_2 = \mathbf{v}_2^T\Gamma\mathbf{v}_2, \quad f_3 = \mathbf{v}_3^T\Gamma\mathbf{v}_3$$

corresponding to the three eigenvectors of Υ. It is not difficult to see that these quantities are invariant with respect to rotation. In fact the quantities are also invariant with respect to translation, to see this consider an arbitrary translation given by T. So in the new coordinates $\Gamma' = \Gamma - T\Upsilon$. Since the eigenvectors $\mathbf{v}_i$ is unchanged by a translation we have,

$$\begin{aligned} f_i' &= \mathbf{v}_i^T\Gamma'\mathbf{v}_i \\ &= \mathbf{v}_i^T\Gamma\mathbf{v}_i - \mathbf{v}_i^T T\Upsilon\mathbf{v}_i \\ &= \mathbf{v}_i^T\Gamma\mathbf{v}_i - \eta_i\mathbf{v}_i^T T\mathbf{v}_i \\ &= \mathbf{v}_i^T\Gamma\mathbf{v}_i \end{aligned}$$

Here η_i is the eigenvalue corresponding to $\mathbf{v}_i$ and the last term vanishes because T is antisymmetric. We will call these invariant the f-invariants.

Now since we can always permute the axes with a rotation, we cannot expect to find these f-invariants themselves as functions of the von Mises invariants. Rather we look for symmetric polynomials in the invariants. Choose a normal form for K with Γ symmetric and Υ diagonal. So the three eigenvectors of Υ lie along the coordinate axes. The f-invariants now comprise the diagonal elements of Γ, $f_i = \mathbf{v}_i^T\Gamma\mathbf{v}_i$. If we write η_1, η_2 and η_3 for the eigenvalues of Υ, that is the diagonal entries of Υ in this coordinate system, then evaluating the determinants in the appendix, we have the following results for some of the von Mises' invariants,

$$\begin{aligned} f_1 + f_2 + f_3 &= \frac{1}{2}b_1 \\ (\eta_2 + \eta_3)f_1 + (\eta_3 + \eta_1)f_2 + (\eta_1 + \eta_2)f_3 &= \frac{1}{2}b_3' \\ \eta_2\eta_3 f_1 + \eta_3\eta_1 f_2 + \eta_1\eta_2 f_3 &= \frac{1}{2}b_5'' \end{aligned}$$

We will look at these invariants again is section 5.1, here we simply note that we can express them in terms of the von Mises' invariants.

3.3 Simplification

In some circumstances it is possible to reduce the stiffness matrix to a simpler form than the Lončarić normal form. For example it may be possible to diagonalise the stiffness matrix using a rigid body transformation. This will only be possible when some particular relation holds amongst the entries of the stiffness matrix. These relations will be independent of the coordinate frame since they express the fact that the matrix can be brought into some particular form by a rigid coordinate transformation. Hence, it must be possible to write the relations as relations among the von Mises' invariants.

Finding the relation is generally hard, it is very similar to the problem of finding conditions for polynomials to factorise. However, in simple cases it is possible.

Suppose we were interested in the conditions under which the stiffness matrix has a centre of compliance. There are several different definitions of a centre of compliance in the literature but here we mean the classical case. That is, a point where small torques produce rotations about the same axis and where forces applied to the point produce translations in the same direction as the forces. Such a stiffness matrix must be transformable to the form,

$$K = \begin{pmatrix} \alpha I_3 & 0 \\ 0 & \beta I_3 \end{pmatrix}$$

for some constants α and β, see [8] for example.

The von Mises' invariants for the simplified matrix are,

$$\det(K - \lambda Q_0 - \mu Q_\infty) = -(\lambda^2 + \mu\beta - \alpha\beta)^3$$

Expanding the right-hand side gives the following polynomial in μ and λ,

$$-\lambda^6 - (3\beta\mu - 3\alpha\beta)\lambda^4 - (3\alpha^2\beta^2 - 6\alpha\beta^2\mu + 3\beta^2\mu^2)\lambda^2 + \alpha^3\beta^3 - 3\alpha^2\beta^3\mu + 3\alpha\beta^3\mu^2 - \beta^3\mu^3$$

From this we can read-off the von Mises' invariants,

$$\begin{array}{lcl@{\qquad}lcl}
b_1 & = & 0 & b_5' & = & 0 \\
b_2 & = & -3\alpha\beta & b_6' & = & 3\alpha^2\beta^3 \\
b_3 & = & 0 & b_5'' & = & 0 \\
b_4 & = & 3\alpha^2\beta^2 & b_6'' & = & 3\alpha\beta^3 \\
b_5 & = & 0 & a_1 & = & 3\beta \\
b_6 & = & \alpha^3\beta^3 & a_2 & = & 3\beta^2 \\
b_3' & = & 0 & a_3 & = & \beta^3 \\
b_4' & = & 6\alpha\beta^2 & & &
\end{array}$$

Now it is a simple matter to derive relations among the von Mises' invariants which are necessary conditions for an arbitrary stiffness matrix to have a centre of compliance. Clearly we must have,

$$b_1 = 0, \quad b_3 = 0, \quad b_5 = 0, \quad b_3' = 0, \quad b_5' = 0, \quad b_5'' = 0$$

and

$$3b_4 - b_2^2 = 0, \quad 27b_6 + b_2^3 = 0, \quad 3b_4' + 2a_1b_2 = 0, \quad 9b_6' - a_1b_2^2 = 0, \quad 9b_6'' + a_1^2b_2 = 0$$

and finally,

$$3a_2 - a_1^2 = 0, \quad 27a_3 - a_1^3 = 0$$

It remains to show that these are also sufficient conditions for the matrix to have a compliance centre. To do this consider the final pair of equations $3a_2 - a_1^2 = 0$ and $27a_3 - a_1^3 = 0$, when satisfied these conditions ensure that the eigenvalues of Υ are all the same and hence that $\Upsilon = \beta I_3$ for some constant β. Now this relation will hold in any coordinate system so we can rotate and translate coordinates without changing this form and transform the stiffness matrix to the Lončarić frame where Γ will be diagonal. It is simple to check that the condition given in section 3.1 for the Lončarić frame to exist is satisfied so long as $\beta \neq 0$.

So let us assume that $\Gamma = \mathrm{diag}(\gamma_{11}, \gamma_{22}, \gamma_{33})$. The relation $b_1 = 0$ now becomes, $\gamma_{11} + \gamma_{22} + \gamma_{33} = 0$. Notice that, if $\Upsilon = \beta I_3$ and $\mathrm{Tr}(\Gamma) = 0$ then the relations $b_3' = 0$ and $b_5'' = 0$ are automatically satisfied. Next we turn to the relation $9b_6'' + a_1^2 b_2 = 0$. We may evaluate this using the form we have so far, that is $\Upsilon = \beta I_3$ and $\Gamma = \mathrm{diag}(\gamma_{11}, \gamma_{22}, \gamma_{33})$. For example, using the result in the appendix we have,

$$b_2 = (\gamma_{11}^2 + \gamma_{22}^2 + \gamma_{33}^2) + 4(\gamma_{11}\gamma_{22} + \gamma_{22}\gamma_{33} + \gamma_{33}\gamma_{11}) - \beta(x_{11} + x_{22} + x_{33})$$

The original relation gives,

$$9b_6'' + a_1^2 b_2 = 36\beta^2(\gamma_{11}\gamma_{22} + \gamma_{22}\gamma_{33} + \gamma_{33}\gamma_{11}) = 0$$

If this relation is satisfied we see that the relation $3b_4' + 2a_1 b_2 = 0$ is automatically satisfied.

The relations $b_3 = 0$ and $b_5' = 0$ can be combined form the relation $3b_5' + a_1 b_3 = 0$ and in terms of our form for the stiffness matrix this becomes,

$$3b_5' + a_1 b_3 = 0 = 24\gamma_{11}\gamma_{22}\gamma_{33} = 0$$

As long as $\beta \neq 0$ the three relations,

$$\gamma_{11} + \gamma_{22} + \gamma_{33} = 0, \quad \gamma_{11}\gamma_{22} + \gamma_{22}\gamma_{33} + \gamma_{33}\gamma_{11} = 0, \quad \gamma_{11}\gamma_{22}\gamma_{33} = 0$$

imply that,

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = 0$$

and hence that $\Gamma = 0$.

Finally, if $\Upsilon = \beta I_3$ and $\Gamma = 0$ the relations, $3b_4 - b_2^2 = 0$ and $27b_6 + b_2^3 = 0$ are simply relations for the rotational invariants of Ξ and, exactly like the relations for Υ, these relations imply that the eigenvalues of Ξ are identical (in this coordinate frame). Hence, $\Xi = \alpha I_3$ for some constant α, and of course this will be true in all coordinate frames.

In conclusion we have shown that the seven relations,

$$\begin{gathered} 3a_2 - a_1^2 = 0, \quad 27a_3 - a_1^3 = 0, \\ b_1 = 0, \quad 9b_6'' + a_1^2 b_2 = 0, \quad 3b_5' + a_1 b_3 = 0, \\ 3b_4 - b_2^2 = 0, \quad 27b_6 + b_2^3 = 0 \end{gathered}$$

are sufficient for the stiffness matrix to have a compliance centre.

4 Ball's Principal Screws

In his famous treatise [1], Ball introduced what he termed the principal screws of potential. The idea behind these screws was the following, suppose the rigid body is in an equilibrium configuration, now we displace it by a small amount given by a screw $\mathbf{s}$ the forces and torques generated by the potential form a wrench. Now for the principal screws of the potential the displacement screw and the wrench will have the same line of action and the same pitch. In the current notation this can be expressed as,

$$K\mathbf{s} = \lambda Q_0 \mathbf{s}$$

Notice that this is essentially an eigenvector problem with the constant λ as eigenvalues. In the modern literature these constants are often called eigenstiffnesses. They are the solutions to the characteristic equation,

$$\det(K - \lambda Q_0) = 0$$

hence we expect six solutions in general and correspondingly six principal screws $\mathbf{s}$ — sometimes called eigenscrews.

4.1 General Properties

Many results on the eigenvectors of a symmetric matrix can be transferred quite simply to the principal screws. However, there are some surprises.

An example of the first case is that principal screws with different eigenstiffnesses are reciprocal, $\mathbf{s}_i^T Q_0 \mathbf{s}_j = 0$. This is just like the fact that the eigenvectors of a symmetric matrix are orthogonal. To prove it we write,

$$K\mathbf{s}_i = \lambda_i Q_0 \mathbf{s}_i, \qquad K\mathbf{s}_j = \lambda_j Q_0 \mathbf{s}_j$$

for the two principal screws. Then we premultiply by the transpose of the other principal screw to get,

$$\mathbf{s}_j^T K\mathbf{s}_i = \lambda_i \mathbf{s}_j^T Q_0 \mathbf{s}_i, \qquad \mathbf{s}_i^T K\mathbf{s}_j = \lambda_j \mathbf{s}_i^T Q_0 \mathbf{s}_j$$

Now since both K and Q_0 are symmetric we have that $\mathbf{s}_j^T K\mathbf{s}_i = \mathbf{s}_i^T K\mathbf{s}_j$ and $\mathbf{s}_j^T Q_0\mathbf{s}_i = \mathbf{s}_i^T Q_0\mathbf{s}_j$. So we can subtract one equation from the other to give,

$$(\lambda_i - \lambda_j)\mathbf{s}_i^T Q_0 \mathbf{s}_j = 0$$

by hypothesis $\lambda_i \neq \lambda_j$ so we must have that,

$$\mathbf{s}_i^T Q_0 \mathbf{s}_j = 0$$

By a very similar argument we also have that

$$\mathbf{s}_i^T K \mathbf{s}_j = 0$$

Ball called screws satisfying this relation "conjugate screws of potential".

A slightly surprising result is that the principal screws don't necessarily exist. Contrast this with the case of symmetric matrices where we are guaranteed a full set of eigenvectors. This is because the eigenvalues are always real. This is not the case here, it is possible that the eigenstiffnesses are complex in which case no real principal screw exists. However, we can learn something by following the proof for the eigenvectors case. We begin by assuming that the eigenstiffness and principal screw are complex, we pair the defining equation for the principal screw with its complex conjugate,

$$\mathbf{s}^\dagger K \mathbf{s} = \lambda \mathbf{s}^\dagger Q_0 \mathbf{s}$$

here † is the Hermitian conjugate, $\mathbf{s}^\dagger = (\mathbf{s}^*)^T$. Now since both K and Q_0 are symmetric and real we have that,

$$\left(\mathbf{s}^\dagger K \mathbf{s}\right)^* = \mathbf{s}^\dagger K \mathbf{s}, \qquad \text{and} \qquad \left(\mathbf{s}^\dagger Q_0 \mathbf{s}\right)^* = \mathbf{s}^\dagger Q_0 \mathbf{s}$$

That is these quantities are real. Now if $\mathbf{s}^\dagger Q_0 \mathbf{s} \neq 0$ we may divide by it and show that λ is real. However, Q_0 is not positive definite and hence $\mathbf{s}^\dagger Q_0 \mathbf{s}$ may vanish. In other words complex principal screws can only be complex lines, $\mathbf{s}^\dagger Q_0 \mathbf{s} = 0$ i.e. pitch 0 screws. Moreover since complex eigenstiffnesses will always appear in complex conjugate pairs the corresponding complex eigenstiffnesses will be a pair of complex conjugate lines.

We will see later that for stiffness matrices corresponding to stable equilibrium configurations a full set of principal screws always exists, that is, they are all real and hence all the eigenstiffnesses are also real.

4.2 Invariants and Expansions

From the definition of the principal screws given above, it is clear that some of the von Mises invariant can be expressed in terms of the eigenstiffnesses,

$$b_1 = \sum_{i=1}^{6} \lambda_i, \quad b_2 = \sum_{1 \le i < j \le 6} \lambda_i \lambda_j, \quad \cdots \quad , \quad b_6 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6$$

In fact these results will always hold since any complex solutions of the characteristic equation for λ will always occur in complex conjugate pairs.

When a full set of eigenvectors do exist we can expand the stiffness matrix as,

$$K = \sum_{i=1}^{6} \frac{\lambda_i}{2\boldsymbol{\omega}_i \cdot \mathbf{v}_i} Q_0 \mathbf{s}_i \mathbf{s}_i^T Q_0$$

where $\mathbf{s}_i^T = (\boldsymbol{\omega}_i^T, \mathbf{v}_i^T)$ is the i-th eigenscrew with eigenstiffness λ_i. This expansion is possible because the eigenscrews are mutually reciprocal. A similar expansion would be always possible if we allowed complex eigenstiffnesses and eigenscrews.

For the partitioned form of the stiffness matrix we can write the expansion as,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} = \sum_{i=1}^{6} \frac{\lambda_i}{2\boldsymbol{\omega}_i \cdot \mathbf{v}_i} \begin{pmatrix} \mathbf{v}_i \mathbf{v}_i^T & \mathbf{v}_i \boldsymbol{\omega}_i^T \\ \boldsymbol{\omega}_i \mathbf{v}_i^T & \boldsymbol{\omega}_i \boldsymbol{\omega}_i^T \end{pmatrix}$$

The pitches of the eigenscrews are also invariant and hence we expect to be able to find expressions for them in terms of the von Mises' invariants. The pitch of the i-th eigenscrew is given by,

$$\pi_i = \frac{\boldsymbol{\omega}_i \cdot \mathbf{v}_i}{\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_i}$$

From the expansion above we can see that,

$$a_1 = \mathrm{Tr}(\Upsilon) = \frac{1}{2}\sum_{i=1}^{6}\frac{\lambda_i}{\pi_i}$$

and also confirm that,

$$b_1 = 2\,\mathrm{Tr}(\Gamma) = \sum_{i=1}^{6}\lambda_i$$

We can also find, $\mathrm{Tr}(\Upsilon\Gamma)$ by observing,

$$\mathrm{Tr}(\Upsilon\Gamma) = \sum_{i=1}^{6}\sum_{j=1}^{6}\frac{\lambda_i\lambda_j}{4(\boldsymbol{\omega}_i\cdot\mathbf{v}_i)(\boldsymbol{\omega}_j\cdot\mathbf{v}_j)}\,\mathrm{Tr}(\boldsymbol{\omega}_i\boldsymbol{\omega}_i^T\mathbf{v}_j\boldsymbol{\omega}_j^T)$$

Now, $\mathrm{Tr}(\boldsymbol{\omega}_i\boldsymbol{\omega}_i^T\mathbf{v}_j\boldsymbol{\omega}_j^T) = \mathrm{Tr}(\boldsymbol{\omega}_j^T\boldsymbol{\omega}_i\boldsymbol{\omega}_i^T\mathbf{v}_j) = (\boldsymbol{\omega}_j\cdot\boldsymbol{\omega}_i)(\boldsymbol{\omega}_i\cdot\mathbf{v}_j)$. When $i \neq j$ we get pairs of terms,

$$\frac{\lambda_i\lambda_j}{4(\boldsymbol{\omega}_i\cdot\mathbf{v}_i)(\boldsymbol{\omega}_j\cdot\mathbf{v}_j)}(\boldsymbol{\omega}_j\cdot\boldsymbol{\omega}_i)(\boldsymbol{\omega}_i\cdot\mathbf{v}_j + \boldsymbol{\omega}_j\cdot\mathbf{v}_k) = 0$$

These terms vanish because different eigenscrews are reciprocal. Hence we have the result,

$$a_1b_1 - b_3' = 2\,\mathrm{Tr}(\Upsilon\Gamma) = \sum_{i=1}^{6}\frac{\lambda_i^2}{2\pi_i}$$

Again, this is a relation among invariants and hence is valid for any coordinate system. Clearly, it would be possible to obtain several similar results by examining invariants such as $\mathrm{Tr}(\Upsilon^2\Gamma)$.

Finally here, we derive a relation between the principal screws and the origin of the Lončarić frame. In the Lončarić frame the submatrix Γ will be symmetric so that,

$$\sum_{i=1}^{6}\frac{\lambda_i}{\boldsymbol{\omega}_i\cdot\mathbf{v}_i}(\mathbf{v}_i\boldsymbol{\omega}_i^T - \boldsymbol{\omega}_i\mathbf{v}_i^T) = 0$$

A short application of the vector triple product rule shows that the matrix, $\mathbf{v}\boldsymbol{\omega}^T - \boldsymbol{\omega}\mathbf{v}^T$ is the 3×3 anti-symmetric matrix corresponding to the vector $\mathbf{v}\times\boldsymbol{\omega}$. So we can write the above relation as,

$$\sum_{i=1}^{6}\frac{\lambda_i}{\boldsymbol{\omega}_i\cdot\mathbf{v}_i}\mathbf{v}_i\times\boldsymbol{\omega}_i = 0$$

Now let us write $\mathbf{v}_i = \mathbf{r}_i\times\boldsymbol{\omega}_i + \pi_i\boldsymbol{\omega}_i$ where $\mathbf{r}_i$ is a point on the axis of the ith principal screw. We can always choose these points so that $\mathbf{r}_i\cdot\boldsymbol{\omega}_i = 0$ and now substituting in the relation above gives,

$$\sum_{i=1}^{6}\frac{\lambda_i}{\pi_i}\mathbf{r}_i = 0$$

With a different choice of origin this relation will not hold. So the origin of the Lončarić has some significance for the principal screws.

4.3 Some Special Cases

Finally here, we look at some special cases for the pitches and dispositions of the eigenscrews.

Is it possible for an eigenscrew to be a pure translation? The equation,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}$$

shows that this can only happen if Υ is singular, $\det(\Upsilon) = 0$. This condition is necessary but not sufficient. We must also have that $\mathbf{v}$ is an eigenvalue of the 3×3 matrix Γ and further that $\mathbf{v}$ lies in the null space of Υ.

We can also ask when can a principal screw have zero pitch? This can be answered in a similar manner by considering the equation,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{0} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\omega} \end{pmatrix}$$

From this we can see that we must have $\boldsymbol{\omega}$ as and eigenvector of Γ^T lying in the null space of Ξ. That is, a principal screw can only be a pure rotation if Ξ is singular.

Next we consider the possibility of parallel principal screws. Let two such screws be,

$$\mathbf{s}_1 = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{r}_1 \times \boldsymbol{\omega} + \pi_1 \boldsymbol{\omega} \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{r}_2 \times \boldsymbol{\omega} + \pi_2 \boldsymbol{\omega} \end{pmatrix}$$

The relation $\mathbf{s}_1^T Q_0 \mathbf{s}_2 = 0$ reduces to $(\pi_1 + \pi_2)\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 0$. Hence this can only happen if the pitches of the principal screws are equal and opposite.

Lastly, can two principal screws have axes which pass through the same point? Consider the screws,

$$\mathbf{s}_1 = \begin{pmatrix} \boldsymbol{\omega}_1 \\ \mathbf{r} \times \boldsymbol{\omega}_1 + \pi_1 \boldsymbol{\omega}_1 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} \boldsymbol{\omega}_2 \\ \mathbf{r} \times \boldsymbol{\omega}_2 + \pi_2 \boldsymbol{\omega}_2 \end{pmatrix}$$

Now the fact that the screws must be reciprocal becomes, $(\pi_1 + \pi_2)\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2 = 0$. So two principal screws which pass through the same point have either orthogonal axes or opposite pitches.

5 Compliance Axes

In [4, 5] Patterson and Lipkin introduced the concepts of force-compliant and torque-compliant axes. These were related to older ideas contained in Dimentberg [10]. The reason for introducing these axes was essentially a practical one, Patterson and Lipkin wanted to generalise structures like the remote centre compliance wrist.

5.1 Force-Compliant Axes

Force-compliant axes satisfies,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \eta \begin{pmatrix} \mathbf{c} \times \mathbf{v} \\ \mathbf{v} \end{pmatrix}$$

a small translation in the direction of the axis produces a pure force along the axis. Notice here, in contrast with the principal screws of Ball, a infinite pitch motion produces a pitch 0 wrench. Clearly $\mathbf{v}$ must be an eigenvector of Υ, so we have three orthogonal candidates. Such an axis exists if and only if we can find a point $\mathbf{c}$ through which the axis passes. We have three linear equations for $\mathbf{c}$,

$$\eta\mathbf{c}\times\mathbf{v} = \Gamma\mathbf{v}$$

these equations are not independent, since $\mathbf{v}\cdot(\mathbf{c}\times\mathbf{v}) = 0$. Hence we have a consistency condition for an eigenvector of Υ to be a force-compliant axis,

$$\mathbf{v}^T\Gamma\mathbf{v} = 0$$

Therefore, we may have no force-compliant axes, 1, 2 or 3 force-compliant axes. Further, if two or all three eigenvectors of Υ coincide then it is possible that any vector in the eigenspace of Υ could be the direction for a force-compliant axis.

In section 3.2, we defined the f-invariants of the stiffness matrix. Here we can see that the vanishing of the the f-invariants determine the number of force-compliant axes. Clearly, if $b_1 = b_3' = b_5'' = 0$ then all the f-invariants vanish and we must have 3 force compliant axes. To find the condition for a single force compliant axis to exist we can solve the equations given in section 3.2 for f_1, f_2 and f_3 and then form the product, $f_1f_2f_3$, certainly this will vanish if and only if one or more of the f-invariants is zero. The solution to the linear equations for the f-invariants is,

$$\begin{aligned} 2\Delta f_1 &= (b_1\eta_1^2 - b_3'\eta_1 + b_5'')(\eta_2-\eta_3) \\ 2\Delta f_2 &= (b_1\eta_2^2 - b_3'\eta_2 + b_5'')(\eta_1-\eta_3) \\ 2\Delta f_3 &= (b_1\eta_3^2 - b_3'\eta_3 + b_5'')(\eta_1-\eta_2) \end{aligned}$$

where the determinant $\Delta = (\eta_1-\eta_2)(\eta_1-\eta_3)(\eta_2-\eta_3)$ and the η_is are the eigenvalues of Υ. Note that if the stiffness matrix has a centre of compliance as in section 3.3, then $\Delta = 0$ and we cannot find the f-invariants this way. However, in such a case it is simple to see that these invariants will all be zero since $\Gamma = 0$.

If we multiply these equations together we get a polynomial in the von Mises' invariants b_1, b_3' and b_5'' whose coefficients are symmetric polynomials in the η_is. These symmetric polynomials in the eigenvalues of Υ can also be written in terms of von Mises invariants since,

$$\begin{aligned} a_1 &= \eta_1+\eta_2+\eta_3 \\ a_2 &= \eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1 \\ a_3 &= \eta_1\eta_2\eta_3 \end{aligned}$$

After a fairly lengthy calculation we obtain the result,

$$\begin{aligned} 8\Delta^2 f_1f_2f_3 = {} & a_3^2b_1^3 - a_3b_3' + b_5''^3 + (a_1^2-a_2)b_1b_5''^2 - a_1b_3'b_5''^2 \\ & + (a_2^2 - a_1a_3)b_1^2b_5'' + a_2b_3'^2b_5'' + a_1a_3b_1b_3'^2 - a_2a_3b_1^2b_5'' - (a_1a_2-3a_3)b_1b_3'b_5'' \end{aligned}$$

When all the eigenvalues of Υ are different the vanishing of this invariant indicates the existence of a force compliant axis.

Now suppose that we had three force-compliant axes. In the coordinates where Υ is diagonal, all the diagonal entries of Γ must vanish. Recall from section 2 above, that this means that we can translate the coordinate frame to a location where Γ is antisymmetric. Say that in this location we have, $\Gamma = P$ where P is the antisymmetric matrix corresponding to a point $\mathbf{p}$. For each of the force compliant axes the point $\frac{1}{\eta_i}\mathbf{p}$

lies on the axis. So the force compliant axes all meet a common line. This doesn't place any restriction on the disposition of the lines, however we can say a little more about $\mathbf{p}$. The equations for the force compliant axes now become,

$$\mathbf{p}\times\mathbf{v}_1=\eta_1\mathbf{c}_1\times\mathbf{v}_1,\quad \mathbf{p}\times\mathbf{v}_2=\eta_2\mathbf{c}_2\times\mathbf{v}_2,\quad \mathbf{p}\times\mathbf{v}_3=\eta_3\mathbf{c}_3\times\mathbf{v}_3$$

and hence we have that,

$$\mathbf{p}-\eta_1\mathbf{c}_1=\alpha_1\mathbf{v}_1,\quad \mathbf{p}-\eta_2\mathbf{c}_2=\alpha_2\mathbf{v}_2,\quad \mathbf{p}-\eta_3\mathbf{c}_3=\alpha_3\mathbf{v}_3$$

where the α_is are constants. Now we can choose the points $\mathbf{c}_i$ on the axes so that their position vectors are normal to the direction of their axes, $\mathbf{c}_i\cdot\mathbf{v}_i=0$. Dotting each of the above equations with the appropriate $\mathbf{v}_i$ and noting that the $\mathbf{v}_i$s form and orthonormal set of unit vectors we have that, $\mathbf{p}=\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\alpha_3\mathbf{v}_3$. Finally we can add the three equations given above to get the relation,

$$\mathbf{p}=\frac{1}{2}(\eta_1\mathbf{c}_1+\eta_2\mathbf{c}_2+\eta_3\mathbf{c}_3)$$

5.2 Torque-Compliant Axes

We can treat torque-compliant axes in a similar fashion. Here a small rotation about an axis produces a pure torque about the axis. The can be written,

$$\begin{pmatrix}\Xi & \Gamma\\ \Gamma^T & \Upsilon\end{pmatrix}\begin{pmatrix}\boldsymbol{\omega}\\ \mathbf{r}\times\boldsymbol{\omega}\end{pmatrix}=\mu\begin{pmatrix}\boldsymbol{\omega}\\ \mathbf{0}\end{pmatrix}$$

Assuming the Υ is non-singular, which is certainly the case in a stable equilibrium configuration. then we can eliminate the term in $\mathbf{r}\times\boldsymbol{\omega}$ to give an eigenvalue equation,

$$\big(\Xi-\Gamma\Upsilon^{-1}\Gamma^T\big)\boldsymbol{\omega}=\mu\boldsymbol{\omega}.$$

So $\boldsymbol{\omega}$ must be an eigenvector of the symmetric matrix $\big(\Xi-\Gamma\Upsilon^{-1}\Gamma^T\big)$. The consistency equation now reads,

$$\boldsymbol{\omega}^T\Upsilon^{-1}\Gamma^T\boldsymbol{\omega}=0$$

Notice that the combination $\big(\Xi-\Gamma\Upsilon^{-1}\Gamma^T\big)$, is invariant with respect to translations and transforms like a standard 3×3 symmetric matrix under rotations. Hence, the three rotational invariants of this matrix, the eigenvalues or equivalently the trace, determinant and the sum of 2×2 principal minors, should be expressible in terms of the von Mises' invariants. Note that these invariants and the rotational invariants of Υ were introduced in [6].

Another way of looking at this is to observe that the definition for the torque-compliant axes can be written as,

$$K\mathbf{s}=\mu Q_\infty\mathbf{s}$$

The solutions to this eigenvalue problem will only be torque-compliant axes if they are lines, that is if they satisfy, $\mathbf{s}^TQ_0\mathbf{s}=0$. The eigenvalues μ are clearly the solutions to the characteristic equation,

$$b_6-b_6'\mu+b_6''\mu^2-a_3\mu^3=0$$

where the coefficients are von Mises' invariants. However, it is clear that solutions to this equations will also be solutions to the equation,

$$\det\left(\Xi - \Gamma\Upsilon^{-1}\Gamma^T - \mu I_3\right) = 0$$

and vice versa. Hence, up to a constant factor the coefficients will be the same. It is not difficult to see that we will have,

$$b_6 = a_3 \det\left(\Xi - \Gamma\Upsilon^{-1}\Gamma^T\right),\ b_6' = a_3 \mathbf{\Theta}\left(\Xi - \Gamma\Upsilon^{-1}\Gamma^T\right),\ b_6'' = a_3 \operatorname{Tr}\left(\Xi - \Gamma\Upsilon^{-1}\Gamma^T\right)$$

where $\mathbf{\Theta}$ denotes the sum of 2×2 principal minors. These results can also be derived by observing that,

$$\begin{pmatrix} \Xi - \mu I_3 & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ -\Upsilon^{-1}\Gamma^T & I_3 \end{pmatrix} = \begin{pmatrix} \Xi - \Gamma\Upsilon^{-1}\Gamma^T - \mu I_3 & \Gamma \\ 0 & \Upsilon \end{pmatrix}$$

Taking determinants we get,

$$\det(K - \mu Q_\infty) = \det(\Upsilon)\det(\Xi - \Gamma\Upsilon^{-1}\Gamma^T - \mu I_3)$$

We have an interesting special case here. Suppose $\Upsilon^{-1}\Gamma^T = -T$ an antisymmetric matrix. Certainly any vector would satisfy the consistency conditions and hence we would have 3 torque compliant axes. However, when this holds we have, $\Gamma - T\Upsilon = 0$, in other words there exists a choice of origin for which $\Gamma' = 0$. In which case the torque-compliant axes are eigenvectors for Ξ' and all pass through the origin.

The definitions of the force and torque-compliant axes may seem a little unsymmetrical at first sight. When the stiffness matrix is non-singular we may invert it to obtain the compliance matrix $C = K^{-1}$, of the system. In terms of the compliance matrix the force compliant axes may be defined as lines $(\mathbf{c} \times \mathbf{v}^T, \mathbf{v}^T)^T$ which satisfy,

$$C \begin{pmatrix} \mathbf{c} \times \mathbf{v} \\ \mathbf{v} \end{pmatrix} = \eta' Q_\infty' \begin{pmatrix} \mathbf{c} \times \mathbf{v} \\ \mathbf{v} \end{pmatrix}$$

Here, $Q_\infty' = Q_0 Q_\infty Q_0$ is the appropriate invariant for compliance matrices. The symmetry of the definitions and their coordinate invariance in now clear.

5.3 Force/Torque-Compliant Axes

Suppose we have a force and a torque-compliant axis which share the same axis. This would mean that the following relations are satisfied simultaneously,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \eta \begin{pmatrix} \mathbf{c} \times \mathbf{v} \\ \mathbf{v} \end{pmatrix}, \qquad \begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{c} \times \mathbf{v} \end{pmatrix} = \mu \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}$$

Now, if we assume that μ and η are both positive, we may take $\pm\sqrt{\mu}$ time the first relation and $\sqrt{\eta}$ times the second to give,

$$\begin{aligned} \begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} \left\{ \pm\sqrt{\mu} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} + \sqrt{\eta} \begin{pmatrix} \mathbf{v} \\ \mathbf{c} \times \mathbf{v} \end{pmatrix} \right\} &= \mu\sqrt{\eta} \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix} \pm \eta\sqrt{\mu} \begin{pmatrix} \mathbf{c} \times \mathbf{v} \\ \mathbf{v} \end{pmatrix} \\ &= \pm\sqrt{\eta\mu} \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \left\{ \pm\sqrt{\mu} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} + \sqrt{\eta} \begin{pmatrix} \mathbf{v} \\ \mathbf{c} \times \mathbf{v} \end{pmatrix} \right\} \end{aligned}$$

The result is a pair of principal screws with exactly the same axis but with opposite eigenstiffnesses and and pitches. Clearly the eigenstiffnesses are $\pm\sqrt{\eta\mu}$ and the pitches are $\pm\sqrt{\mu/\eta}$.

We shall see in the following section that if the stiffness matrix refers to a stable equilibrium position then the eigenvalues μ and η certainly will be positive. But of course, in general, we are not guaranteed such a configuration of compliance axes.

Notice that the argument given above is completely reversible. So that if we have a pair of eigenstiffnesses on the same axis with equal but opposite eigenstiffness and pitches then there will be a force-compliant axis and a torque compliant axis on the same line.

Suppose we had a stiffness matrix with a pair of force/torque compliant axes. Let us write the corresponding principal screws as,

$$\mathbf{s}_1^{\pm} = \begin{pmatrix} \sqrt{\eta_1}\mathbf{v}_1 \\ \pm\sqrt{\mu_1}\mathbf{v}_1 + \sqrt{\eta_1}\mathbf{c}_1 \times \mathbf{v}_1 \end{pmatrix}, \qquad \mathbf{s}_2^{\pm} = \begin{pmatrix} \sqrt{\eta_2}\mathbf{v}_2 \\ \pm\sqrt{\mu_2}\mathbf{v}_2 + \sqrt{\eta_2}\mathbf{c}_2 \times \mathbf{v}_2 \end{pmatrix}$$

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal the fact that the principal screws are reciprocal reduces to,

$$(\mathbf{s}_1^{\pm})^T Q_0 \mathbf{s}_2^{\pm} = 2\sqrt{\eta_1\eta_2}(\mathbf{c}_1 - \mathbf{c}_2)\cdot(\mathbf{v}_1 \times \mathbf{v}_2) = 0$$

Now, assuming that neither η_1 nor η_2 are zero, this means that a line joining any pair of points on the axes of the two principal screws must be perpendicular to the vector perpendicular to both axes. This can only happen if the axes are coplanar and hence in the axes meet at a point.

When we have three force/torque axes, each pair of principal screws must have coplanar axes and hence all three axes will meet at a common point. By choosing the origin to be at this common intersection and aligning the coordinate axes with the mutually perpendicular force/torque axes we see that in this coordinate frame the stiffness matrix will be diagonal. Moreover, it is simple to show that if the stiffness matrix is diagonal then it will have three force/torque axes, hence this property completely characterises the diagonalisable stiffness matrices.

6 Stability

In general, we are usually most interested in configurations of the rigid body which are stable. That is to say, local minima of the potential functions. For stable equilibria we can say a little more about the structure of the stiffness matrix. Most of the ideas in this section can be found in Patterson and Lipkin [4, 5].

Another interpretation of stability here is that to move the body a small amount from equilibrium requires work to be done on the body. Hence, we must have,

$$\mathbf{s}^T K \mathbf{s} > 0$$

for all screws $\mathbf{s}$.

An immediate application of this is to the compliance axes we met in the last section. If $\mathbf{s}^T = (\mathbf{0}^T,\, \mathbf{v}^T)$ is a force compliance axis of a stable stiffness matrix then,

$$(\mathbf{0}^T,\, \mathbf{v}^T)K\begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \eta(\mathbf{0}^T,\, \mathbf{v}^T)\begin{pmatrix} \mathbf{c}\times\mathbf{v} \\ \mathbf{v} \end{pmatrix} = \eta|\mathbf{v}|^2$$

Since this must be positive we must have that $\eta > 0$. Similarly a torque-compliant axis for a stable stiffness matrix must have a positive eigenvalue μ.

Another consequence of the above is that, for a stable stiffness matrix the submatrices Ξ and Υ must be positive definite. This means that the Lončarić invariant is never zero,

$$L = \det(\mathrm{Tr}(\Upsilon)I_3 - \Upsilon) \neq 0$$

This could only vanish if Υ had a pair of eigenvalues equal in magnitude but with opposite signs. Hence, a stable stiffness matrix can always be transformed to Lončarić normal form.

By Sylvester's law of inertia, since K is positive definite we can find a non-singular matrix M which diagonalises K. That is, $M^TKM = I_6$. So the characteristic equation for the eigenstiffnesses can be written,

$$0 = \det(M^T)\det(K - \lambda Q_0)\det(M) = \det(I_6 - \lambda M^T Q_0 M)$$

This means that the eigenstiffnesses are the inverses of the standard eigenvalues of the matrix M^TQ_0M. This is a non-singular symmetric matrix, hence all it's eigenvalues are real. Moreover, a short calculation reveals that Q_0 has rank 6 and semi-index 0, that is, it has 3 positive and 3 negative eigenvalues. The rank and semi-index of a matrix are invariant with respect to non-singular congruences, thus we may infer that a stiffness matrix K is stable if and only if it has 3 positive and 3 negative eigenstiffnesses.

The result depends on the existence of a matrix M, however, this matrix is not invariant with respect to rigid changes, in different coordinate frames the matrix M, which diagonalises K, will be different. But this does not affect the argument.

Now, since we can use the expansion of the stiffness matrix found in section 4.2,

$$\begin{pmatrix} \Xi & \Gamma \\ \Gamma^T & \Upsilon \end{pmatrix} = \sum_{i=1}^{6} \frac{\lambda_i}{2\boldsymbol{\omega}_i \cdot \mathbf{v}_i} \begin{pmatrix} \mathbf{v}_i\mathbf{v}_i^T & \mathbf{v}_i\boldsymbol{\omega}_i^T \\ \boldsymbol{\omega}_i\mathbf{v}_i^T & \boldsymbol{\omega}_i\boldsymbol{\omega}_i^T \end{pmatrix}$$

Our condition for stability must hold for the eigenscrews and hence we have,

$$\mathbf{s}_i^T K \mathbf{s}_i = \lambda_i \mathbf{s}_i^T Q_0 \mathbf{s}_i = 2\lambda_i \pi_i |\boldsymbol{\omega}_i|^2 > 0$$

From this we see that the pitch of an eigenscrew cannot be zero or infinite ($\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_i = 0$) for a stable stiffness matrix. Further, since $|\boldsymbol{\omega}_i|^2$ is always positive, we see that π_i must have the same sign as λ_i and thus for a stable stiffness matrix we must have three eigenscrews with positive pitches (right-handed) and 3 with negative pitches (left-handed).

7 Conclusions

In this work we have looked at several ways of studying the stiffness matrix for a rigid body at equilibrium in an arbitrary potential field. This is a huge subject and we have only been able to give a few results and examples. However, we hope that this is enough to give a flavour of the subject and possibly stimulate further work.

Many workers in this area also consider the compliance matrix of the system. A dual theory will exist for this matrix, however, in the case where the stiffness matrix is non-singular the compliance matrix is simply the inverse of the stiffness matrix. Hence all results for the compliance matrix can be derived from the stiffness matrix, nevertheless some problems are more neatly expressed in terms of the compliance matrix of the system.

There are still many outstanding problems here. For example, suppose we had a pair of stiffness matrices with exactly the same set of von Mises's invariants, is there always a rigid change of coordinates which will transform one matrix into the other?

Another example is the precise relationship between Ball's principal screws and the von Mises' invariants. We have given a few results in this direction but in general the eigenstiffnesses, the pitches of the principal screws and their disposition are invariants of the stiffness matrix and hence it should be possible to find the von Mises' invariants in terms of these invariants.

In general, it is the covariants of the stiffness matrix as well as the invariants that we should be interested in. Covariants are vectors and matrices whose entries are polynomial functions of the stiffness matrix's entries and which transform according to some representation of the group of rigid body motions. Clearly, many covariants can be constructed from the principal screws and compliance axes. However we have already seen an example in the 3×3 symmetric matrix $\Xi - \Gamma\Upsilon^{-1}\Gamma^T$.

One of the difficulties in this work is the large amount of symbolic computation that is required. Modern computer algebra programs should be able to make short work of this and hence we expect substantial progress in the near future.

One problem that we have not addressed at all here is the problem of designing systems with a given stiffness matrix. There are several works in the literature where this is solved for systems of simple springs, see [11]. We hope that this work will inform a slightly different problem: What should the stiffness matrix be for the system we want to design? This is essentially the view taken in [4, 5], Patterson and Lipkin claim that their force-compliant and torque-compliant axes are important for practical systems. Once the desired stiffness matrix has been chosen we can set about the problem of realising it using springs and so forth.

However, there is another application for this approach in robotics. Rather than build passive systems we could implement the stiffness using the control system of the robot. This is the idea behind impedance control. Current impedance control methods do not usually take account of the complexity of the stiffness matrix, often the desired stiffness matrix is simply chosen to be diagonal. Of course there is another problem here, we need a damping matrix!

References

[1] R.S. Ball. *The Theory of Screws*. Cambridge University Press, Cambridge, 1900.

[2] R. von Mises. Motorrechnung, ein neues hilfsmittel in der mechanik. *Zeitschrift für Angewandte Mathematik und Mechanik*, 4(2):155–181, 1924. English Translation by E.J. Baker and K. Wohlhart. Motor Calculus, a New Theoretical Device for Mechanics. Institute for Mechanics, University of Technology Graz, Austria, 1996.

[3] J. Lončarić. Normal forms of stiffness and compliance matrices. *IEEE J. of Robotics and Automation*, RA-3(6):567–572, 1987.

[4] T. Patterson and H. Lipkin. Structure of robot compliance. *J. Mechanical Design*, 115:576–580, 1993.

[5] T. Patterson and H. Lipkin. A classification of robot compliance. *J. Mechanical Design*, 115:581–584, 1993.

[6] J.W. Burdick Q. Lin and E. Rimon. A quality measure for compliant grasps and fixtures using stiffness characterization. *IEEE Trans. on Robotics and Automation*, 16(6):675–688, 2000.

[7] S. Demy H. Bruyninckx and V. Kumar. Generalized stability of compliant grasps. In *IEEE Conference on Robotics and Automation, Leuven, Belgium*, pages 2396–2402. IEEE, 1998.

[8] J.M. Selig. The spatial stiffness matrix from simple stretched springs. In *IEEE Conference on Robotics and Automation, San Francisco*, pages 3314–3319. IEEE, 2000.

[9] J.M. Selig. *Geometrical Methods in Robotics*. Springer Verlag, New York, 1996.

[10] F.M. Dimentberg. *Screw Calculus and its Applications in Mechanics*. Izd. Nauka, Moscow, 1965. English translation, Foreign Technology Division, Wright-Patterson Air Force Base, Ohio, Document No. FTD-HT-23-1632-67, 1968.

[11] R.G. Roberts. Minimal realization of spatial stiffness matrix with simple springs connected in parallel. *IEEE Trans. Robotics and Automation*, 15(5):953–958, 1999.

8 Appendix — Von Mises' Invariants in Terms of Determinants

The first result here serves to set up our notation.

$$b_6 = \begin{vmatrix} x_{11} & x_{12} & x_{13} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{12} & x_{22} & x_{23} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{13} & x_{23} & x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{21} & \gamma_{31} & u_{11} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & u_{12} & u_{22} & u_{23} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} = \det(K) = \det(\Upsilon)\det\left(\Xi - \Gamma\Upsilon^{-1}\Gamma^T\right)$$

$$b_5 = 2\begin{vmatrix} x_{12} & x_{13} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{22} & x_{23} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{23} & x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{22} & \gamma_{32} & u_{12} & u_{22} & u_{23} \\ \gamma_{23} & \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} + 2\begin{vmatrix} x_{11} & x_{13} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{12} & x_{23} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{13} & x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{31} & u_{11} & u_{12} & u_{13} \\ \gamma_{13} & \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} + 2\begin{vmatrix} x_{11} & x_{12} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{12} & x_{22} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{13} & x_{23} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{21} & u_{11} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{12} & u_{22} & u_{23} \end{vmatrix}$$

$$\begin{aligned} b_4 = {} & \begin{vmatrix} x_{11} & x_{12} & \gamma_{11} & \gamma_{12} \\ x_{12} & x_{22} & \gamma_{21} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & u_{11} & u_{12} \\ \gamma_{12} & \gamma_{22} & u_{12} & u_{22} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{13} & \gamma_{11} & \gamma_{13} \\ x_{13} & x_{33} & \gamma_{31} & \gamma_{33} \\ \gamma_{11} & \gamma_{31} & u_{11} & u_{13} \\ \gamma_{13} & \gamma_{33} & u_{13} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{22} & x_{23} & \gamma_{22} & \gamma_{23} \\ x_{23} & x_{33} & \gamma_{32} & \gamma_{33} \\ \gamma_{22} & \gamma_{32} & u_{22} & u_{23} \\ \gamma_{23} & \gamma_{33} & u_{23} & u_{33} \end{vmatrix} + \\ & \begin{vmatrix} x_{12} & x_{13} & \gamma_{11} & \gamma_{13} \\ x_{23} & x_{33} & \gamma_{31} & \gamma_{33} \\ \gamma_{22} & \gamma_{32} & u_{12} & u_{23} \\ \gamma_{23} & \gamma_{33} & u_{13} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{13} & \gamma_{11} & \gamma_{12} \\ x_{12} & x_{23} & \gamma_{21} & \gamma_{22} \\ \gamma_{11} & \gamma_{31} & u_{11} & u_{12} \\ \gamma_{13} & \gamma_{33} & u_{13} & u_{23} \end{vmatrix} + \begin{vmatrix} x_{12} & x_{13} & \gamma_{11} & \gamma_{12} \\ x_{22} & x_{23} & \gamma_{21} & \gamma_{22} \\ \gamma_{22} & \gamma_{32} & u_{12} & u_{22} \\ \gamma_{23} & \gamma_{33} & u_{13} & u_{23} \end{vmatrix} + \\ & \begin{vmatrix} x_{12} & x_{23} & \gamma_{22} & \gamma_{23} \\ x_{13} & x_{33} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{31} & u_{12} & u_{13} \\ \gamma_{13} & \gamma_{33} & u_{23} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & \gamma_{11} & \gamma_{13} \\ x_{13} & x_{23} & \gamma_{31} & \gamma_{33} \\ \gamma_{11} & \gamma_{21} & u_{11} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{12} & u_{23} \end{vmatrix} + \begin{vmatrix} x_{12} & x_{22} & \gamma_{22} & \gamma_{23} \\ x_{13} & x_{23} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{21} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{22} & u_{23} \end{vmatrix} - \\ & \begin{vmatrix} x_{11} & x_{12} & x_{13} & \gamma_{11} \\ \gamma_{11} & \gamma_{21} & \gamma_{31} & u_{11} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & u_{12} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & u_{13} \end{vmatrix} - \begin{vmatrix} x_{12} & x_{22} & x_{23} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & \gamma_{31} & u_{12} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & u_{22} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & u_{23} \end{vmatrix} - \begin{vmatrix} x_{13} & x_{23} & x_{33} & \gamma_{33} \\ \gamma_{11} & \gamma_{21} & \gamma_{31} & u_{13} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & u_{23} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & u_{33} \end{vmatrix} - \\ & \begin{vmatrix} x_{11} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{12} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{13} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & u_{11} & u_{12} & u_{13} \end{vmatrix} - \begin{vmatrix} x_{12} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{22} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{23} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{22} & u_{12} & u_{22} & u_{23} \end{vmatrix} - \begin{vmatrix} x_{13} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{23} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
b_3 &= 2\begin{vmatrix} x_{12} & x_{13} & \gamma_{11} \\ \gamma_{22} & \gamma_{32} & u_{12} \\ \gamma_{23} & \gamma_{33} & u_{13} \end{vmatrix} + 2\begin{vmatrix} x_{11} & x_{13} & \gamma_{11} \\ \gamma_{11} & \gamma_{31} & u_{11} \\ \gamma_{13} & \gamma_{33} & u_{13} \end{vmatrix} + 2\begin{vmatrix} x_{11} & x_{12} & \gamma_{11} \\ \gamma_{11} & \gamma_{21} & u_{11} \\ \gamma_{12} & \gamma_{22} & u_{12} \end{vmatrix} + \\
&\quad 2\begin{vmatrix} x_{22} & x_{23} & \gamma_{22} \\ \gamma_{22} & \gamma_{32} & u_{22} \\ \gamma_{23} & \gamma_{33} & u_{23} \end{vmatrix} + 2\begin{vmatrix} x_{12} & x_{23} & \gamma_{22} \\ \gamma_{11} & \gamma_{31} & u_{12} \\ \gamma_{13} & \gamma_{33} & u_{23} \end{vmatrix} + 2\begin{vmatrix} x_{12} & x_{22} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & u_{12} \\ \gamma_{12} & \gamma_{22} & u_{22} \end{vmatrix} + \\
&\quad 2\begin{vmatrix} x_{23} & x_{33} & \gamma_{33} \\ \gamma_{22} & \gamma_{32} & u_{23} \\ \gamma_{23} & \gamma_{33} & u_{33} \end{vmatrix} + 2\begin{vmatrix} x_{13} & x_{33} & \gamma_{33} \\ \gamma_{11} & \gamma_{31} & u_{13} \\ \gamma_{13} & \gamma_{33} & u_{33} \end{vmatrix} + 2\begin{vmatrix} x_{13} & x_{23} & \gamma_{33} \\ \gamma_{11} & \gamma_{21} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{23} \end{vmatrix} + 2\begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} \\
b_2 &= 2\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} + 2\begin{vmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{vmatrix} + 2\begin{vmatrix} \gamma_{22} & \gamma_{23} \\ \gamma_{32} & \gamma_{33} \end{vmatrix} - \\
&\quad 2\begin{vmatrix} x_{12} & \gamma_{11} \\ \gamma_{22} & u_{12} \end{vmatrix} - 2\begin{vmatrix} x_{13} & \gamma_{11} \\ \gamma_{33} & u_{13} \end{vmatrix} - 2\begin{vmatrix} x_{23} & \gamma_{22} \\ \gamma_{33} & u_{23} \end{vmatrix} - \\
&\quad \begin{vmatrix} x_{11} & \gamma_{11} \\ \gamma_{11} & u_{11} \end{vmatrix} - \begin{vmatrix} x_{22} & \gamma_{22} \\ \gamma_{22} & u_{22} \end{vmatrix} - \begin{vmatrix} x_{33} & \gamma_{33} \\ \gamma_{33} & u_{33} \end{vmatrix} \\
b_1 &= 2\gamma_{11} + 2\gamma_{22} + 2\gamma_{33} = 2\,\mathrm{Tr}(\Gamma) \\
b_6' &= \begin{vmatrix} x_{22} & x_{23} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ x_{23} & x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{21} & \gamma_{31} & u_{11} & u_{12} & u_{13} \\ \gamma_{22} & \gamma_{32} & u_{12} & u_{22} & u_{23} \\ \gamma_{23} & \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{13} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{13} & x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{31} & u_{11} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{32} & u_{12} & u_{22} & u_{23} \\ \gamma_{13} & \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ x_{12} & x_{22} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{11} & \gamma_{21} & u_{11} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{12} & u_{22} & u_{23} \\ \gamma_{13} & \gamma_{23} & u_{13} & u_{23} & u_{33} \end{vmatrix} \\
&= \det(\Upsilon)\boldsymbol{\Theta}(\Xi - \Gamma\Upsilon^{-1}\Gamma^T) \\
b_6'' &= \begin{vmatrix} x_{11} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{11} & u_{11} & u_{12} & u_{13} \\ \gamma_{12} & u_{12} & u_{22} & u_{23} \\ \gamma_{13} & u_{13} & u_{23} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{22} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{21} & u_{11} & u_{12} & u_{13} \\ \gamma_{22} & u_{12} & u_{22} & u_{23} \\ \gamma_{23} & u_{13} & u_{23} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \gamma_{31} & u_{11} & u_{12} & u_{13} \\ \gamma_{32} & u_{12} & u_{22} & u_{23} \\ \gamma_{33} & u_{13} & u_{23} & u_{33} \end{vmatrix} \\
&= \det(\Upsilon)\,\mathrm{Tr}\left(\Xi - \Gamma\Upsilon^{-1}\Gamma^T\right) \\
b_5' &= 2\begin{vmatrix} x_{23} & x_{33} & \gamma_{31} & \gamma_{33} \\ \gamma_{21} & \gamma_{31} & u_{11} & u_{13} \\ \gamma_{22} & \gamma_{32} & u_{12} & u_{23} \\ \gamma_{23} & \gamma_{33} & u_{13} & u_{33} \end{vmatrix} + 2\begin{vmatrix} x_{22} & x_{23} & \gamma_{21} & \gamma_{22} \\ \gamma_{21} & \gamma_{31} & u_{11} & u_{12} \\ \gamma_{22} & \gamma_{32} & u_{12} & u_{22} \\ \gamma_{23} & \gamma_{33} & u_{13} & u_{23} \end{vmatrix} - 2\begin{vmatrix} x_{13} & x_{33} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & \gamma_{31} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{32} & u_{22} & u_{23} \\ \gamma_{13} & \gamma_{33} & u_{23} & u_{33} \end{vmatrix} + \\
&\quad 2\begin{vmatrix} x_{11} & x_{13} & \gamma_{11} & \gamma_{12} \\ \gamma_{11} & \gamma_{31} & u_{11} & u_{12} \\ \gamma_{12} & \gamma_{32} & u_{12} & u_{22} \\ \gamma_{13} & \gamma_{33} & u_{13} & u_{23} \end{vmatrix} - 2\begin{vmatrix} x_{12} & x_{22} & \gamma_{22} & \gamma_{23} \\ \gamma_{11} & \gamma_{21} & u_{12} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{22} & u_{23} \\ \gamma_{13} & \gamma_{23} & u_{23} & u_{33} \end{vmatrix} - 2\begin{vmatrix} x_{11} & x_{12} & \gamma_{11} & \gamma_{13} \\ \gamma_{11} & \gamma_{21} & u_{11} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{12} & u_{23} \\ \gamma_{13} & \gamma_{23} & u_{13} & u_{33} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
b_5'' &= 2\begin{vmatrix} \gamma_{11} & u_{12} & u_{13} \\ \gamma_{12} & u_{22} & u_{23} \\ \gamma_{13} & u_{23} & u_{33} \end{vmatrix} - 2\begin{vmatrix} \gamma_{21} & u_{11} & u_{13} \\ \gamma_{22} & u_{12} & u_{23} \\ \gamma_{23} & u_{13} & u_{33} \end{vmatrix} + 2\begin{vmatrix} \gamma_{31} & u_{11} & u_{12} \\ \gamma_{32} & u_{12} & u_{22} \\ \gamma_{33} & u_{13} & u_{23} \end{vmatrix} \\
&= 2\,\mathrm{Tr}(\Upsilon^2\Gamma) + \mathrm{Tr}(\Upsilon)^2\,\mathrm{Tr}(\Gamma) - 2\,\mathrm{Tr}(\Upsilon)\,\mathrm{Tr}(\Upsilon\Gamma) - \mathrm{Tr}(\Upsilon^2)\,\mathrm{Tr}(\Gamma) \\
b_4' &= \begin{vmatrix} x_{33} & \gamma_{31} & \gamma_{33} \\ \gamma_{31} & u_{11} & u_{13} \\ \gamma_{33} & u_{13} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{22} & \gamma_{21} & \gamma_{22} \\ \gamma_{21} & u_{11} & u_{12} \\ \gamma_{22} & u_{12} & u_{22} \end{vmatrix} + \begin{vmatrix} x_{33} & \gamma_{32} & \gamma_{33} \\ \gamma_{32} & u_{22} & u_{23} \\ \gamma_{33} & u_{23} & u_{33} \end{vmatrix} + \\
&\quad \begin{vmatrix} x_{11} & \gamma_{11} & \gamma_{12} \\ \gamma_{11} & u_{11} & u_{12} \\ \gamma_{12} & u_{12} & u_{22} \end{vmatrix} + \begin{vmatrix} x_{22} & \gamma_{22} & \gamma_{23} \\ \gamma_{22} & u_{22} & u_{23} \\ \gamma_{23} & u_{23} & u_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & \gamma_{11} & \gamma_{13} \\ \gamma_{11} & u_{11} & u_{13} \\ \gamma_{13} & u_{13} & u_{33} \end{vmatrix} + \\
&\quad 2\begin{vmatrix} x_{23} & \gamma_{31} & \gamma_{33} \\ \gamma_{21} & u_{11} & u_{13} \\ \gamma_{22} & u_{12} & u_{23} \end{vmatrix} - 2\begin{vmatrix} x_{13} & \gamma_{32} & \gamma_{33} \\ \gamma_{11} & u_{12} & u_{13} \\ \gamma_{12} & u_{22} & u_{23} \end{vmatrix} + 2\begin{vmatrix} x_{12} & \gamma_{22} & \gamma_{23} \\ \gamma_{11} & u_{12} & u_{13} \\ \gamma_{13} & u_{23} & u_{33} \end{vmatrix} - \\
&\quad 2\begin{vmatrix} \gamma_{21} & \gamma_{31} & u_{11} \\ \gamma_{22} & \gamma_{32} & u_{12} \\ \gamma_{23} & \gamma_{33} & u_{13} \end{vmatrix} + 2\begin{vmatrix} \gamma_{11} & \gamma_{31} & u_{12} \\ \gamma_{12} & \gamma_{32} & u_{22} \\ \gamma_{13} & \gamma_{33} & u_{23} \end{vmatrix} - 2\begin{vmatrix} \gamma_{11} & \gamma_{21} & u_{13} \\ \gamma_{12} & \gamma_{22} & u_{23} \\ \gamma_{13} & \gamma_{23} & u_{33} \end{vmatrix} \\
b_3' &= 2\begin{vmatrix} \gamma_{11} & \gamma_{12} \\ u_{12} & u_{22} \end{vmatrix} + 2\begin{vmatrix} \gamma_{22} & \gamma_{23} \\ u_{23} & u_{33} \end{vmatrix} + 2\begin{vmatrix} \gamma_{11} & \gamma_{13} \\ u_{13} & u_{33} \end{vmatrix} - 2\begin{vmatrix} \gamma_{31} & \gamma_{33} \\ u_{11} & u_{13} \end{vmatrix} - 2\begin{vmatrix} \gamma_{21} & \gamma_{22} \\ u_{11} & u_{12} \end{vmatrix} - 2\begin{vmatrix} \gamma_{32} & \gamma_{33} \\ u_{22} & u_{23} \end{vmatrix} \\
&= 2\,\mathrm{Tr}(\Upsilon)\,\mathrm{Tr}(\Gamma) - 2\,\mathrm{Tr}(\Upsilon\Gamma) \\
a_1 &= u_{11} + u_{22} + u_{33} = \mathrm{Tr}(\Upsilon) \\
a_2 &= \begin{vmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{vmatrix} + \begin{vmatrix} u_{11} & u_{13} \\ u_{13} & u_{33} \end{vmatrix} + \begin{vmatrix} u_{22} & u_{23} \\ u_{23} & u_{33} \end{vmatrix} = \boldsymbol{\Theta}(\Upsilon) = \tfrac{1}{2}(\mathrm{Tr}(\Upsilon^2) - \mathrm{Tr}(\Upsilon)^2) \\
a_3 &= \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{vmatrix} = \det(\Upsilon)
\end{aligned}
$$