On the Plane Symmetric Bricard Mechanism

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Abstract. In this paper the motion of the plane symmetric Bricard 6R mechanism is studied. A simple method based on the dimensions of intersecting varieties in the Study quadric is used to show that the mechanism is mobile. The degree of the motion of the third link (the one adjacent to the plane of symmetry) relative to the plane of symmetry is found. The degree and genus of the motion of the third link relative to the first link is also found. This curve in the Study quadric is given as the intersection of the variety generated by the RR dyad formed by second and third joints with the variety of displacements that keep the fourth joint axis in the special linear line complex whose axis is the axis of the first joint. Finally, the motion of the symmetry plane when the second link is fixed is considered. The symmetry planes comprise the common tangent planes to a pair of circularly symmetric hyperboloids.

Keywords: Plane symmetry, overconstrained mechanisms, elliptic curves

1 Introduction

At the end of the 19th century Bricard discovered several types of mobile 6R mechanisms, [2]. In this work one of these types, the plane symmetric 6R, will be studied in some detail. Many other workers have looked at this mechanism previously, see for example [3] and [7] and references therein. The focus in this work is a little different. The idea is to study the mechanism using simple geometrical techniques and then use knowledge from Algebraic geometry to give very general results concerning the degree and genus of various curves associated to the mechanism.

The mechanism under consideration consists of six revolute joints arranged in a closed loop. The axes of the first and fourth joints lie in a plane Π, the fifth and sixth joint axes are the reflections in Π of the third and second joint axes respectively, see Figure 1.

Notice that, with this arrangement the axes of the second and sixth joint will meet at a point on Π, as will the axes of the third and fifth joint. The line joining these two intersection point must lie in Π and hence will meet (or be coplanar with) the axes of first and fourth joints. Hence, all six joints lie in a special line complex and are thus linearly dependant.

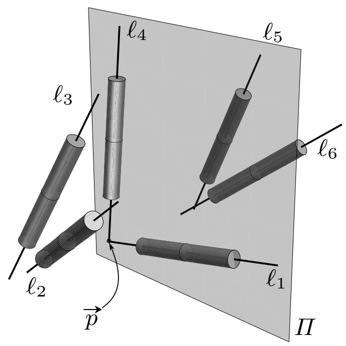

Fig. 1. The Arrangement of Joints in the Plane Symmetric Bricard Mechanism. Note that, for clarity, the links are not shown here.

2 Line in a Plane

The first task is to show that the mechanism is indeed mobile. Suppose we fix the plane Π and the axis of the first joint in Π. If the mechanism is mobile then the fourth joint will move but can only move in the plane Π. So we are led to consider the set of rigid-body displacements that can move a line in such a way that it remains in a fixed plane. This displacement subvariety will be a four dimensional subvariety of Study quadric, call it X for the moment. It may be parametriesed as a rotation about the line, composed with planar displacements in Π. This can be thought of as a Segre variety $X = \mathbb{P}^1 \times \mathbb{P}^3$. To be specific, let Π be the xz-plane and assume that the fourth joint axis is the x-axis. That is, $\ell_4 = i$. The rigid-body motions that maintain ℓ_4 in the plane Π can be parameterised using dual quatrnions as,

$$\begin{aligned} g &= (\alpha_0 + \alpha_1 j + \alpha_2 \varepsilon i + \alpha_3 \varepsilon k)(c + si) \\ &= c\alpha_0 + s\alpha_0 i + c\alpha_1 j - s\alpha_1 k - s\alpha_2 \varepsilon + c\alpha_2 \varepsilon i + s\alpha_3 \varepsilon j + c\alpha_3 \varepsilon k \end{aligned}$$

This can be written as,

$$\begin{aligned} a_0 &= c\alpha_0 \quad , & c_0 &= -s\alpha_2, \\ a_1 &= s\alpha_0, & c_1 &= c\alpha_2, \\ a_2 &= c\alpha_1, & c_2 &= s\alpha_3, \\ a_3 &= -s\alpha_1, & c_3 &= c\alpha_3. \end{aligned}$$

In general, a $\mathbb{P}^3 \times \mathbb{P}^1$ Segre variety is defined by 6 degree 2 equations. These equations represent quadric hypersurfaces in $\mathbb{P}^7$ and can be expressed as requiring the following matrix to have rank 1,

$$\text{Rank} \begin{pmatrix} a_0 & a_2 & c_1 & c_3 \\ a_1 & -a_3 & -c_0 & c_2 \end{pmatrix} = 1.$$

The six quadrics are given by the vanishing of the 2×2 sub-determinants of the matrix,

$$\begin{aligned} Q_1 &= a_0a_3 + a_1a_2 = 0, & Q_4 &= a_2c_0 - a_3c_1 = 0,\\ Q_2 &= a_0c_0 + a_1c_1 = 0, & Q_5 &= a_2c_2 + a_3c_3 = 0,\\ Q_3 &= a_0c_2 - a_1c_3 = 0, & Q_6 &= c_1c_2 + c_0c_3 = 0. \end{aligned}$$

Notice that, the Study quadric lies in the linear system formed by these quadrics, $Q_S = Q_2 + Q_5$. The degree of this Segre variety is known to be $\binom{3+1}{1} = 4$, [6].

Intersecting this subvariety with the 3-dimensional subvariety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ generated by the first three joints of the mechanism will produce a one-dimensional variety in the Study quadric. This curve represents a rigid-body motion that can be followed by the final link of the open loop chain formed from the mechanism's first three joints and links. Since the rigid motions keeps the axis of the fourth joint in the plane Π, the reflection of the first four joints in Π will produce the same motion of ℓ_4.

The above argument demonstrates the mobility of the mechanism since the intersection is a curve. The degree of the motion of the third link as a curve in the Study quadric can also be found. In [10] it was shown that the set of rigid-body displacements that maintain a given line in a fixed linear line complex is the intersection of the Study quadric with another quadric hypersurface in $\mathbb{P}^7$. Now consider a pair of lines ℓ_a and ℓ_b, both in Π and meeting at a point $\vec{p}$. The displacement variety of group elements that maintain ℓ_4 in the special linear line complex with axis ℓ_a will be denoted Q_a and similarly Q_b for the variety of displacements that maintain ℓ_4 in the special line complex determined by ℓ_b. Recall, that a special linear line complex is the set of all lines meeting or parallel to the axis of the complex. Now the intersection of these three quadrics $Q_S \cap Q_a \cap Q_b$, is clearly a four dimensional variety that contains the variety we are interested in $X \subset Q_S \cap Q_a \cap Q_b$. As a four-dimensional subvariety of $\mathbb{P}^7$, $Q_S \cap Q_a \cap Q_b$ is a complete intersection and hence will have degree 8. However, as a subvariety of the Study quadric it has homology class $4\sigma_4$, where σ_4 is the generator for homology in dimension 4, see [11]. Now X is only one component of this variety, there is another component. The other component of $Q_S \cap Q_a \cap Q_b$ consists of the group elements that maintain the incidence between ℓ_4 and $\vec{p}$, the intersection point of ℓ_a and ℓ_b. This component can also be seen to be isomorphic to the Segre variety $\mathbb{P}^3 \times \mathbb{P}^1$, call it Y say. So $Q_S \cap Q_a \cap Q_b = X \cup Y$, both X and Y have degree 4 and hence their homology class in Q_s must both be $2\sigma_4$.

Next look again at the first three joints, they form a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Segre variety, This variety has homology class $2\sigma_A + 4\sigma_B$ in the Study quadric, see [9]. So if we intersect the two varieties X and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the homology of the resulting curve will be,

$$2\sigma_4 \cap (2\sigma_A + 4\sigma_B) = (2 \times 2 + 2 \times 4)\sigma_1 = 12\sigma_1.$$

In dimension 1, the homology class σ_1 coincides with the degree and so we can conclude that the degree of this motion will be 12 in general.

There is a special case to consider. Suppose the first three joins of the mechanism, (and hence the final three also), have the design parameters of a Bennett linkage, see [9]. In this case the variety of displacements that the third link can perform is given by the complete intersection of the Study quadric with another quadric hypersurface and a $\mathbb{P}^5$. The homology of this subvariety is thus $2\sigma_A + 2\sigma_B$ and so the degree of the motion followed by the third link, relative to the symmetry plane is,

$$2\sigma_4 \cap (2\sigma_A + 2\sigma_B) = (2 \times 2 + 2 \times 2)\sigma_1 = 8\sigma_1.$$

That is, the motion is represented by a degree 8 curve in the Study quadric.

Finally here, notice that the varieties X and Y introduced above can be generated by mechanical linkages, a spherical joint and a prismatic joint for Y and a planar (E) joint linked to a revolute joint for X. For the last case here, the axis of the revolute joint must be parallel to the plane of the planar joint.

3 Fixed First Link

In the previous section the motion of the third link of the mechanism was found relative to the fixed plane Π. It is more usual to look at the relative motion of one link relative to another link, most commonly, the motion of a link relative to the opposite link in the loop. Here, the motion of the third link relative to the first will be investigated.

Fixing the first link means fixing the first and second joints. The plane Π can now rotate about the first joint. The fourth joint must still remain in Π but now as Π moves we can see that ℓ_4 will remain in the special line complex with axis ℓ_1. Note that this motion can be mechanically generated by a UPU linkage, see [8]. As mentioned in the previous section, the variety of rigid-body motions that keep a line in a fixed linear line complex is given by the intersection of the Study quadric with another quadric hypersurface.

On the other hand, the motion of the third link must also lie in the displacement subvariety generated by the second and third joints. This is an RR dyad and from [11], the group elements produced by such a linkage are given by the intersection of the Study quadric with a $\mathbb{P}^3$. So the motion of the third link relative to the first is given by the intersection of a pair of quadrics with a $\mathbb{P}^3$. Intersecting each quadric with the $\mathbb{P}^3$ gives a 2-dimensional quadric lying in the $\mathbb{P}^3$. The intersection of two quadrics in a $\mathbb{P}^3$ is, in general, an elliptic quartic curve, [6].

To investigate this more closely, the lines $\ell_1, \dots \ell_4$ will be written as dual quaternions as,

$$\ell_i = \omega_i + \varepsilon v_i = (\omega_{ix} i + \omega_{iy} j + \omega_{iz} k) + \varepsilon(v_{ix} i + v_{iy} j + v_{iz} k).$$

As 8-component vectors the lines will be written in partitioned form as,

$$\boldsymbol{\ell}_i = \begin{pmatrix} 0 \\ \vec{\omega}_i \\ 0 \\ \vec{v}_i \end{pmatrix}.$$

Now from [10], the quadric defining the displacement variety that keeps ℓ_4 in the special linear line complex defined by ℓ_1 can be written as,

$$\mathbf{g}^T Q \mathbf{g} = 0$$

where $\mathbf{g} = (a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)^T$ is the column vector of homogeneous coordinates in $\mathbb{P}^7$. In partitioned form the 8×8 symmetric matrix Q can be written as,

$$Q = \begin{pmatrix} \Xi & \Upsilon \\ \Upsilon & 0 \end{pmatrix}$$

where,

$$\Xi = \begin{pmatrix} 0 & (\vec{\omega}_1 \times \vec{v}_4 + \vec{v}_1 \times \vec{\omega}_4)^T \\ (\vec{\omega}_1 \times \vec{v}_4 + \vec{v}_1 \times \vec{\omega}_4) & V_4\Omega_1 + \Omega_1 V_4 + \Omega_4 V_1 + V_1 \Omega_4 \end{pmatrix},$$

and

$$\Upsilon = \begin{pmatrix} 0 & (\vec{\omega}_1 \times \vec{\omega}_4)^T \\ (\vec{\omega}_1 \times \vec{\omega}_4) & \Omega_1\Omega_4 + \Omega_4\Omega_1 \end{pmatrix},$$

where, Ω_i is the 3×3 anti-symmetric matrix corresponding to the vector $\vec{\omega}_i$ and V_i corresponds to the vector $\vec{v}_i$, see [10].

The motion of the final link of the RR dyad can be parametrised by the sines and cosines of the joint angles

$$g = (\cos\frac{\theta_2}{2} + \sin\frac{\theta_2}{2}\ell_2)(\cos\frac{\theta_3}{2} + \sin\frac{\theta_3}{2}\ell_3).$$

If we abbreviate $\cos\frac{\theta_i}{2}$ to c_i and $\sin\frac{\theta_2}{2}$ to s_i, the above equation can be expanded to,

$$g = c_2c_3 + c_2s_3\ell_3 + s_2c_3\ell_2 + s_2s_3\ell_2\ell_3.$$

The parametrisation can be written in partitioned 8-vector form as,

$$\mathbf{g} = \begin{pmatrix} c_2c_3 - s_2s_3(\vec{\omega}_2 \cdot \vec{\omega}_3) \\ s_2c_3\vec{\omega}_2 + c_2s_3\vec{\omega}_3 + s_2s_3\vec{\omega}_2 \times \vec{\omega}_3 \\ s_2s_3(\vec{\omega}_2 \cdot \vec{v}_3 + \vec{v}_2 \cdot \vec{\omega}_3) \\ s_2c_3\vec{v}_2 + c_2s_3\vec{v}_3 + s_2s_2(\vec{\omega}_2 \times \vec{v}_3 + \vec{v}_2 \times \vec{\omega}_3) \end{pmatrix}$$

The variety determined by the parameterisation satisfies the equation for the Study quadric and the 3-plane mentioned above. So substituting this into the quadric above, $\mathbf{g}^T Q \mathbf{g} = 0$, will give an equation for the elliptic quartic curve:

$$\begin{aligned}
&\big(2\mathbf{1}^T Q\boldsymbol{\ell}_3 c_3 s_3 + \boldsymbol{\ell}_3^T Q \boldsymbol{\ell}_3 s_3^2\big)c_2^2 \\
&\quad + 2\big(\mathbf{1}^T Q \boldsymbol{\ell}_2 c_3^2 + (\mathbf{1}^T Q \boldsymbol{\ell}_2\boldsymbol{\ell}_3 + \boldsymbol{\ell}_2^T Q \boldsymbol{\ell}_3)c_3s_3 + \boldsymbol{\ell}_3^T Q \boldsymbol{\ell}_2\boldsymbol{\ell}_3 s_3^2\big)c_2s_2 \\
&\qquad \big(\boldsymbol{\ell}_2^T Q \boldsymbol{\ell}_2 c_3^2 + 2\boldsymbol{\ell}_2^T Q \boldsymbol{\ell}_2\boldsymbol{\ell}_3 c_3s_3 + (\boldsymbol{\ell}_2\boldsymbol{\ell}_3)^T Q \boldsymbol{\ell}_2\boldsymbol{\ell}_3 s_3^2\big)s_2^2 = 0,
\end{aligned}$$

where $\mathbf{1}$ is the 8-vector corresponding to the dual quaternion 1. Notice that, $\mathbf{1}^T Q \mathbf{1} = 0$. Now if we divide this equation through by $s_2^2 s_3^2$ and then replace c_2/s_2 by λ and c_3/s_3 by μ, the above equation can be written in the form,

$$a(\mu)\lambda^2 + 2b(\mu)\lambda + c(\mu) = 0,$$

where

$$\begin{aligned}
a(\mu) &= 2\mathbf{1}^T Q\boldsymbol{\ell}_3\mu + \boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_3,\\
b(\mu) &= \mathbf{1}^T Q\boldsymbol{\ell}_2\mu^2 + (\mathbf{1}^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3 + \boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_3)\mu + \boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3,\\
c(\mu) &= \boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_2\mu^2 + 2\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\mu + (\boldsymbol{\ell}_2\boldsymbol{\ell}_3)^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3.
\end{aligned}$$

In general, it is possible to solve for λ in terms of μ by using the familiar formula for solving quadratic equations. The appearance of a square root in the solution confirms that the curve is not rational in general. However, if the discriminant of the quadratic vanishes identically, then we do obtain a rational parameterisation of the curve. A rational quartic curve in $\mathbb{P}^3$ must have a singularity, hence the condition for the curve to be rational is the same as the condition for it to have a singularity. This is a very similar situation to the Grashof condition for planar 4-bar mechanisms, see [4]. The discriminant of the quadratic is a quartic in μ,

$$b(\mu)^2 - a(\mu)c(\mu) = \Delta_4\mu^4 + \Delta_3\mu^3 + \Delta_2\mu^2 + \Delta_1\mu + \Delta_0,$$

The coefficients are functions of the design parameters of the mechanism,

$$\begin{aligned}
\Delta_4 &= \Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\Big)^2,\\
\Delta_3 &= 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\Big)\Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big) + 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\Big)\Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_3\Big) - 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_3\Big)\Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_2\Big),\\
\Delta_2 &= 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\Big)\Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big) + \Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big)^2 + 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big)\Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_3\Big)\\
&\quad + \Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_3\Big)^2 - 4\Big(\mathbf{1}^T Q\boldsymbol{\ell}_3\Big)\Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big) - \Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_2\Big)\Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_3\Big),\\
\Delta_1 &= 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big)\Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big) + 2\Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_3\Big)\Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big)\\
&\quad - 2\Big(\mathbf{1}^T Q\boldsymbol{\ell}_3\Big)\Big((\boldsymbol{\ell}_2\boldsymbol{\ell}_3)^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big) - 2\Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_3\Big)\Big(\boldsymbol{\ell}_2^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big),\\
\Delta_0 &= \Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big)^2 - \Big(\boldsymbol{\ell}_3^T Q\boldsymbol{\ell}_3\Big)\Big((\boldsymbol{\ell}_2\boldsymbol{\ell}_3)^T Q\boldsymbol{\ell}_2\boldsymbol{\ell}_3\Big).
\end{aligned}$$

Where, for example, $\mathbf{1}^T Q\boldsymbol{\ell}_2 = \vec{\omega}_2 \cdot (\vec{\omega}_1 \times \vec{v}_4) + \vec{\omega}_2 \cdot (\vec{v}_1 \times \vec{\omega}_4) + \vec{v}_2 \cdot (\vec{\omega}_1 \times \vec{\omega}_4)$. The geometric meaning of this term vanishing can be found by considering coordinates with origin at $\vec{p}$, the common point of the two lines ℓ_1 and ℓ_4 in Π. By choosing this point as origin both $\vec{v}_1 = \vec{v}_4 = \vec{0}$ and the condition becomes, $\vec{v}_2 \cdot (\vec{\omega}_1 \times \vec{\omega}_4) = 0$. Now, since $\vec{\omega}_1 \times \vec{\omega}_4$ is normal to Π we can see that the condition implies that the plane defined by the line ℓ_2 and $\vec{p}$ must be normal to Π.

Space precludes a full analysis of the five conditions above. Note, however, that singularity is a precondition for the elliptic quartic to degenerate into two or more rational components. The possible degenerations of this mechanism have been recently studied in [3] and [7].

A little more can be said about the case where the discriminant does not vanish identically. In general, the discriminant is a quartic in the variable μ. Such a quartic can have 0, 2 or 4 real roots. Solving for λ in terms of μ gives a map from the configuration curve of the mechanism to θ_2, or more precisely $\cot(\theta_2/2)$. Away from the roots of the discriminant, this map is clearly 2-to-1. Recalling that the configuration curve is an elliptic quartic, Harnack's theorem indicates that the curve can have up to 2 real ovals. That is, the curve can have one or two disjoint real components. In terms of the plane symmetric mechanism this means that it may have one or two disjoint modes of operation. Since the modes are disjoint, to get from one mode to the other the mechanism will have to be disassembled and reassembled in the other mode. Hence these are often refered to as assembly modes of the mechanism. This phenomenon occurs in planar and spherical 4-bar mechanism where both 1 assembly mode and 2 assembly mode mechanisms are possible, see [4]. If the discriminant has four real roots then the configuration curve must have two real ovals and moreover the second joint, directed along ℓ_2 will not rotate fully in either assembly mode, such a joint is usually called a rocker. When the discriminant has only 2 real roots the configuration curve has just one assembly mode and the second joint is still a rocker. If the discriminant doesn't have any real roots, then the second joint must be a crank, meaning that it is fully rotatable. However, in this case extra information is required to tell whether the curve has one or two assembly modes. More details on these ideas can be found in [5], but essentially the roots of the discriminant correspond to ramification, or branch points of the map from the configuration curve to the $\mathbb{P}^1$ determined by the second joint angle. Note also, the intersection of quadric surfaces in space is also of interest in Computer Graphics and CAD and the above analysis has also been considered in [12].

4 Second Link Fixed

Although the mechanism is plane symmetric so far the possibility that one of the links performs a plane symmetric motion has not been considered. Fixing the second link of the mechanism it is clear that fifth link will perform a plane symmetric motion, the reflection of the second link in Π. In this section the motion of Π as the mechanism moves will be considered.

If the second link is fixed then the second and third joint axes are fixed. So the axes of the first joint ℓ_1 will rotate about the axis of the second joint to form a regulus of a one-sheeted hyperboloid of revolution. Likewise, ℓ_4 forms such a regulus about ℓ_3. Now, as the mechanism moves the symmetry plane Π will contain both joint axes ℓ_1 and ℓ_4. If a plane contains a line from a regulus of a hyperbolid then it must also contain a line from the other regulus on the hyperboloid, so that the intersection has degree 2. Hence, such a plane must be tangent to the hyperboloid at some point. So we can see that, as the mechanism moves the plane Π is always a common tangent to the two hyperboloids formed by the lines ℓ_1 and ℓ_4. To study planes in $\mathbb{P}^3$ it is usual to consider plane as point in a dual $\mathbb{P}^3$. The tangent planes to a quadric surface form a quadric surface in the dual $\mathbb{P}^3$. If the quadric surface is given by an equation of the form,

$$\vec{r}^T M \vec{r} = 0,$$

where M is a symmetric 4×4 matrix and $\vec{r}^T = (x,\, y,\, z,\, w)$ are the homogeneous coordinates of $\mathbb{P}^3$. Then the equation satisfied by the "points" in the dual $\mathbb{P}^3$ will be,

$$\left(\vec{q}^{\,*}\right)^T \mathrm{Adj}(M)\,\vec{q}^{\,*} = 0,$$

where $\vec{q}^{\,*}$ are the coordinates of a plane in the dual space and $\mathrm{Adj}(M)$; the adjugate matrix of M.

The common tangent planes to a pair of hyperboloids can be seen to be represented by the intersection of a pair of quadric surfaces in the dual $\mathbb{P}^3$. As above, this is, in general an elliptic quartic curve.

5 Conclusion

The above is of necessity, rather sketchy. Nevertheless the work contains some observations which would appear to be novel. In particular, the identification of the configuration curve of the mechanism as an elliptic quartic curve and the determination of the motion of the symmetry plane as an elliptic quartic in the dual $\mathbb{P}^3$. These ideas prompt several questions about these mechanisms; are there designs of plane symmetric Bricard 6Rs with 2 assembly modes? The 1-parameter family of planes determined by Π as the mechanism moves will be the envelope of a plane curve, what is the degree of this curve and how does it lie in relation to the mechanism?

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