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Research paper

Mechanisms generating line trajectories

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ABSTRACT

In this work the problem of designing mechanisms to guide a line along a ruled surface is considered. This is motivated by the idea of illustrating, by means of a physical model, the set of lines in a general linear line complex. Cylindrical symmetry implies that the problem reduces of designing a mechanism to move a line along the regulus of a rectangular hyperbolic paraboloid.

This leads to the study of several simple ruled surfaces and congruences of lines. In particular the paper studies the regulus of the elliptical hyperboloid, the cylindroid and linear line congruences. In each case relatively simple mechanisms are designed to constrain a line in the mechanism to a given space of lines.

1. Introduction

Over the years much attention has been paid to the design of mechanisms to produce a given point trajectory. Far less emphasis has been given to the design of mechanisms that move a line along a desired trajectory. Indeed the authors could find no mention of this problem in the literature of the past 120 years or so. In [1], "kinematically generated" ruled surfaces are considered. The application considered there was to 5-axis cylindrical milling and laser machining. However, no particular mechanisms were designed as the focus was on producing ruled surfaces that could be followed by general purpose machines.

Of course, the subject of line geometry is central to kinematics, see for example the standard texts, [2–4] where various sets of lines are mentioned in connection with the kinematics of mechanisms and machines. The Mathematics of line geometry has also been used profitably in many other areas, in particular Mechanical Gearing, [5,6], Computer Aided Design [7], Robotics and Geometric Optics.

In [8, p. 37] Felix Klein refers to, and gives a sketch of, a mechanical model produced by Martin Schilling. This model was used to illustrate the linear line complex. The details of this mechanism seem to be lost. So, in part, this work is an attempt to recreate this mechanism. Since the set of lines in the complex has a cylindrical symmetry, see Section 2, we only need to worry about the radial motion of the line. In Section 2.1 it is shown that the radial lines in the general linear line complex form the regulus of a rectangular hyperbolic paraboloid. So the first problem addressed in this work is to design a mechanism to carry a line along this ruled surface.

A simple solution would be to represent the line by a rod that moves in a slotted cylinder, the shape of the slot determined by quartic curve found in Section 2.2. However, the intention here is to design a mechanism using standard elements, that is a mechanism consisting of only lower Reuleaux pair joints. This is facilitated by a knowledge of the set of rigid-body displacements that constrain a line to lie in a general linear complex. A constraint variety first studied by Blaschke, see [9].

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This leads to the study of other simple ruled surfaces, in particular the regulus of a general elliptic hyperboloid and the cylindroid. Straightforward mechanism designs are given to guide a line in the mechanism through the lines of the given ruled surfaces.

Finally, we consider the problem of designing 2 degree-of-freedom mechanisms to move a line in linear congruences, elliptic, hyperbolic or parabolic.

We begin by setting-up some notation that will be used below. Further details of the notation can be found in [10], for example.

Dual quaternions can be used to represent elements of the group of rigid-body displacements, $SE(3)$. A dual quaternion is an expression of the form,

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k).$$

The constants $a_0, \dots, c_3$ are sometimes called the Study parameters or Study coordinates of the element and i, j and k are the unit quaternions in the coordinate directions satisfying the famous relations due to Hamilton: $i^2 = j^2 = k^2 = ijk = -1$. The nilpotent ε, satisfies $\varepsilon^2 = 0$ and commutes with all other elements of the algebra. The dual quaternion conjugate is given by reversing the sign of the generating quaternions i, j and k, so that,

$$g^- = (a_0 - a_1 i - a_2 j - a_3 k) + \varepsilon(c_0 - c_1 i - c_2 j - c_3 k).$$

Dual quaternions which satisfy the relation,

$$gg^- = 1$$

form a Lie group which double covers $SE(3)$, the group of rigid-body displacements of space. That is, both g and $-g$ correspond to the same displacement. To represent the group $SE(3)$ itself we must think of the Study parameters as homogeneous coordinates in a 7-dimensional projective space $\mathbb{P}^7$. The set of group elements now lie on a 6-dimensional projective variety of degree two, know as the Study quadric. The homogeneous equation defining this variety is,

$$a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3 = 0.$$

Points in the Study quadric are in 1-to-1 correspondence with the elements of $SE(3)$ with the exception of a 3-dimensional plane of points satisfying the linear equations $a_0 = a_1 = a_2 = a_3 = 0$. This plane of ideal elements will be referred to as A_∞.

Lines in space can be written as dual quaternions using the Plücker coordinates of the line. In this formalism, a line takes the form,

$$\ell = (\omega_x i + \omega_y j + \omega_z k) + \varepsilon(v_x i + v_y j + v_z k),$$

where, $\boldsymbol{\omega}^T = (\omega_x, \omega_y, \omega_z)$ is a vector in the direction of the line and $\mathbf{v}^T = (v_x, v_y, v_z)$ is the moment of the line about the origin. A rotation with angle θ about the line ℓ is then given by the dual quaternion,

$$g(\theta) = \cos(\theta/2) + \sin(\theta/2)\ell.$$

The Plücker coordinates of a line satisfies the relation,

$$\boldsymbol{\omega} \cdot \mathbf{v} = 0.$$

This can be written as a matrix product,

$$(\omega_x, \omega_y, \omega_z, v_x, v_y, v_z)\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_x \\ v_y \\ v_z \end{pmatrix}$$
$$= (\boldsymbol{\omega}^T, \mathbf{v}^T)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0.$$

The 6×6 matrix above, will be written as Q_0. In partitioned form this matrix is,

$$Q_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

The reciprocal product of a pair of lines is then given by,

$$\ell_1^T Q_0 \ell_2 = (\boldsymbol{\omega}_1^T, \mathbf{v}_1^T)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\omega}_2 \\ \mathbf{v}_2 \end{pmatrix} = \boldsymbol{\omega}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \boldsymbol{\omega}_2.$$

It can be shown that lines are coplanar if and only if their reciprocal product vanishes.

2. The general linear line complex

The set of lines that satisfy a linear equation in the Plücker coordinates form what is known as a linear complex of lines. In the past these sets of lines were also known as screws.

The linear equation for the lines in the complex are,

$$\tau_x\omega_x + \tau_y\omega_y + \tau_z\omega_z + F_x v_x + F_y v_y + F_z v_z = 0.$$

Collecting the constant coefficients $\tau_x, \dots, F_z$ into a column vector gives a wrench,

$$\mathcal{W} = \begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \\ F_x \\ F_y \\ F_z \end{pmatrix}.$$

However, this wrench can also be written as a twist $\mathbf{s}$, multiplied by the matrix, Q_0. That is, $\mathcal{W} = Q_0\mathbf{s}$, and the equation for the complex can be written,

$$\mathbf{s}^T Q_0 \ell = 0.$$

Multiplying the twist $\mathbf{s}$, by an arbitrary non-zero constant clearly does not affect the solution set to this homogeneous equation. Thus the twist is really a screw, hence justifying the identification of linear line complexes with screws.

Any property of a screw is therefore also a property of the corresponding linear line complex. In particular, line complexes have an axis and a pitch. By choosing suitable coordinates a general screw $\mathbf{s}$ can be put into the standard form,

$$\mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ p \end{pmatrix},$$

that is, with the axis of the screw aligned with the z-axis and where the pitch of the screw is p. The set of lines in the complex defined by $\mathbf{s}$ has a cylindrical symmetry about the screw axis. This can be seen by observing that the matrix Q_0 is invariant under the action of the adjoint representation of $SE(3)$, for any group element g, we have, $\mathrm{Ad}(g)^T Q_0 \,\mathrm{Ad}(g) = Q_0$. Hence if c is in the cylindrical subgroup that preserves $\mathbf{s}$, that is $\mathrm{Ad}(c)\mathbf{s} = \mathbf{s}$, then,

$$0 = \mathbf{s}^T Q_0 \ell = \mathbf{s}^T \,\mathrm{Ad}(c)^T Q_0 \,\mathrm{Ad}(c)\ell = \mathbf{s}^T Q_0 \,\mathrm{Ad}(c)\ell.$$

Hence, $\mathrm{Ad}(c)\ell$, the transform of the line ℓ by c, lies in the complex if ℓ is in the complex.

This means that, to understand the disposition of lines in the complex we only need to work out how the lines change as we move away from the screw axis radially. So, consider a line whose closest point to the screw axis lies along the y-axis a distance d from the screw axis. Assume that the angle that the line makes with the x-axis is θ so that the Plücker coordinates of the line are,

$$\ell(\theta) = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \\ d\sin\theta \\ 0 \\ -d\cos\theta \end{pmatrix} \tag{1}$$

see Fig. 1. Substituting this into the linear equation for the complex gives,

$$\mathbf{s}^T Q_0 \ell = p\sin\theta - d\cos\theta = 0.$$

This means that as the line moves away from the screw axis the distance from the axis and angle to the axis are coupled according to the relation,

$$d = p\tan\theta.$$

When $\theta = 0$ and $d = 0$, the line coincides with the x-axis, but as the angle approaches $\pm\pi/2$ the distance increases without limit, that is, the further from the axis the line is the more nearly it is parallel to the axis of the complex. See Fig. 2.

Note that, all the lines in the complex are of this form or can be found from one of these lines by rotating about, and translating along the axis of the complex.

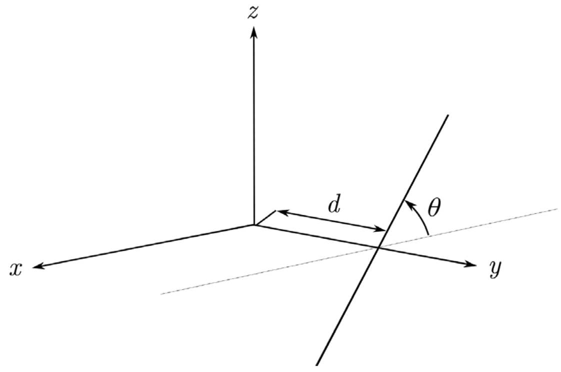

Fig. 1. A line in the line complex perpendicular to the y-axis.

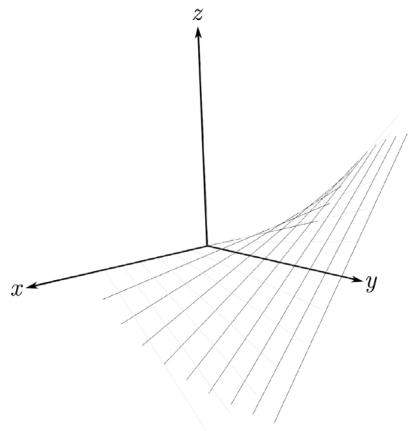

Fig. 2. The reguli of lines in a rectangular hyperbolic paraboloid. The heavy lines are the lines in the complex perpendicular to the y-axis.

2.1. The ruled surface of lines along a radius

The one-parameter family of lines defined by Eq. (1) form a ruled surface. Points on this surface have the form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \lambda\cos\theta \\ p\tan\theta \\ \lambda\sin\theta \end{pmatrix},$$

where λ is an arbitrary parameter. Eliminating the parameters, it can be see that,

$$pz/x = y,$$

or, if we introduce the homogenising variable w,

$$xy - pwz = 0.$$

That is, the ruled surface is a regulus of a rectangular hyperbolic paraboloid.

Such a surface has two rulings, the one given above and the other given by the lines,

$$\ell'(\phi) = \begin{pmatrix} 0 \\ \cos\phi \\ \sin\phi \\ 0 \\ -l\sin\phi \\ l\cos\phi \end{pmatrix},$$

where $l = p\tan\phi$. Notice that, taking the reciprocal product of any pair of lines, one from each regulus gives,

$$\ell(\theta)^T Q_0 \ell'(\phi) = p\sin\theta\sin\phi - p\sin\phi\sin\theta = 0,$$

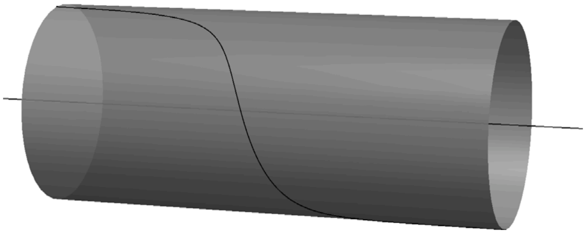

Fig. 3. The quartic curve on a cylinder.

that is, all the lines from one regulus are reciprocal to all the lines from the other regulus. This shows that all the lines from one regulus meet all the lines from the other regulus. In Fig. 2 the grey lines belong to the second regulus on the surface.

2.2. The motion generating the regulus

The regulus of lines described in the previous section can be produced by subjecting an initial line to a rigid-body motion. The regulus can be given by subjecting the x-axis to the motion,

$$M(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & p\tan\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

In general, the trajectories of points in space under this motion are given by,

$$X = x\cos\theta + z\sin\theta,$$
$$Y = y + p\tan\theta,$$
$$Z = -x\sin\theta + z\cos\theta,$$

Thinking of this as a curve in projective space $\mathbb{P}^3$, by introducing the homogenising variable w, and writing $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$, the equations for the trajectories become,

$$X = x(c^2 - s^2)^2 + 2zcs(c^2 - s^2),$$
$$Y = y(c^2 + s^2)(c^2 - s^2) + 2pcs(c^2 + s^2).$$
$$Z = -2xcs(c^2 - s^2) + z(c^2 - s^2)^2,$$
$$W = w(c^2 + s^2)(c^2 - s^2).$$

Here, $(c^2 + s^2) = 1$ has been used to render the equations homogeneous. Treating c and s as homogeneous parameters shows that the trajectories are generally degree 4 curves (quartic curves). Moreover, it can be seen that the curves satisfy $X^2 + Z^2 = x^2 + z^2$, so these trajectories lie on concentric cylinders about the y-axis. See Fig. 3.

2.3. The motion as a curve in the study quadric

The corresponding curve in the Study quadric can be found by mapping the matrix in Eq. (2) to the Study quadric using the mapping given in [9, §3.3] for example. After cancelling common factors the result is,

$$g(c, s) = \big(c(c^2 - s^2) + s(c^2 - s^2)j\big) + \varepsilon p\big(cs^2 - c^2 sj\big). \tag{3}$$

Notice, this is a twisted cubic curve, that is a rational cubic. The cubic intersects the 3-plane of unphysical elements A_∞ in a pair of points when the parameters satisfy $c = \pm s$.

In general, a degree n curve in the Study quadric will act on points in space to produce trajectories with degree $2n$. However, the degree can be reduced depending on how the curve in the Study quadric meets the 3-plane A_∞, see [11].

A twisted cubic curve lies on the intersection of three quadrics in a $\mathbb{P}^3$. In this case these varieties are easy to find. One of the quadrics is the Study quadric of course. The 3-plane is given by the linear equations $a_1 = a_3 = c_1 = c_3 = 0$. The intersection of the 3-plane with the Study quadric gives the set of group elements producing rotations and translations about the y-axis, that is a subgroup of $SE(3)$ comprising of cylindrical displacements about the y-axis. Intersecting this 3-plane with the Study quadric gives a 2-dimensional quadric with equation,

$$a_0c_0 + a_2c_2 = 0.$$

The second quadric can be taken as the subvariety of rigid-body displacements that maintain a line in a fixed line complex. In [9, §5.4] it was shown that the rigid-body displacements that maintain a line $\ell^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$, in a complex determined by the screw $\mathbf{s}^T = (\mathbf{w}^T, \mathbf{u}^T)$, is a quadric in $\mathbb{P}^7$. The equation for the quadric is most easily stated using an 8×8 partitioned matrix,

$$g^T Q g = \begin{pmatrix} a^T & c^T \end{pmatrix} \begin{pmatrix} \Xi & Y \\ Y & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = 0. \tag{4}$$

where $g^T = (\mathbf{a}^T, \mathbf{c}^T) = (a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)^T$ represents the homogeneous coordinates of a group element in $\mathbb{P}^7$. The two 4×4 symmetric matrices Ξ and Y are given in turn by,

$$\Xi = \begin{pmatrix} 0 & (\boldsymbol{\omega} \times \mathbf{u} + \mathbf{v} \times \mathbf{w})^T \\ (\boldsymbol{\omega} \times \mathbf{u} + \mathbf{v} \times \mathbf{w}) & U\Omega + \Omega U + WV + VW \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & (\boldsymbol{\omega} \times \mathbf{w})^T \\ (\boldsymbol{\omega} \times \mathbf{w}) & \Omega W + W\Omega \end{pmatrix}.$$

The matrices, U, Ω and so forth, represent the 3×3 anti-symmetric matrices corresponding to $\mathbf{u}$, $\boldsymbol{\omega}$ *etc*. With $\ell(0)$ and $\mathbf{s}$ as given above, the matrix representing the quadric is,

$$Q_1 = \begin{pmatrix} 0 & 0 & -p & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & 1 \\ -p & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

That is, the quadric is given by the equation,

$$p(a_1a_3 - a_0a_2) - (a_0c_2 - a_1c_3 + a_2c_0 - a_3c_1) = 0.$$

Intersecting this with the 3-plane of cylindrical displacements gives the 2-dimensional quadric with the equation,

$$pa_0a_2 + a_0c_2 + a_2c_0 = 0.$$

To find the third quadric we can inspect the parameterisation of the curve given in Eq. (3) and see that the third quadric is,

$$p(a_0c_0 - a_2c_2) + 2(c_0^2 - c_2^2) = 0.$$

Of course, linear combinations of the 3 quadrics can also be used.

2.4. The twist velocity

The twist velocity of the motion can be found using (2),

$$S = \dot{M}M^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p\sec^2\theta \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a 6-component vector the twist velocity S can thus be written,

$$\mathbf{s} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ p\sec^2\theta \\ 0 \end{pmatrix}.$$

This shows that the axis of the twist is always aligned with the y-axis. The pitch of the twist is not constant, it changes as the line moves away from the axis of the complex. Writing h for the pitch of the motion we have that,

$$h = p\sec^2\theta = \frac{1}{p}(p^2 + d^2),$$

since the distance d of translation along the y-axis satisfies $d = p\tan\theta$.

The fixed axode of the motion is just the axis of the twist velocity. Clearly in this case the fixed axode degenerates to a single line, the y-axis.

The moving axode can be found from the axis of the twist velocity in the moving frame. That is the axis of $M^{-1}\dot{M}$. This gives the same result as the in the globally fixed frame and hence the moving axode also degenerates to a single line.

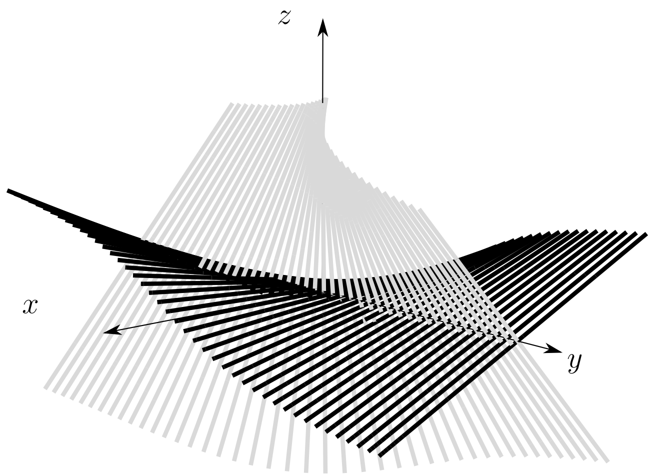

Fig. 4. Zindler's conoid.

2.5. Line-symmetry

Since the motion is about a fixed axis, the motion is line-symmetric. This results from an elementary theorem due to Krames [12, p. 395] (see also [13]).

The ruled surface that acts as the base for this motion can be found in several ways, we could employ the methods outlined in [13] for example. But in this case it is perhaps easiest to use simple geometry. So consider a half-turn that moves the x axis to the line $\ell(\theta)$ given in (1). The axis of this π rotation must cross the y-axis at right angles half way between the x-axis and the line $\ell(\theta)$. That is, it must pass through the point $(0, d/2, 0)$ and lie parallel to the xz-plane. Next, it must bisect the angle between the x-axis and $\ell(\theta)$. However, there are two solutions for this, either the angle from the x-axis to the axis of the half-turn could be $\theta/2$ or $\theta/2 + \pi/2$. That is the axis could have Plücker coordinates,

$$\begin{pmatrix} \cos(\theta/2) \\ 0 \\ \sin(\theta/2) \\ \frac{d}{2}\sin(\theta/2) \\ 0 \\ -\frac{d}{2}\cos(\theta/2) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\sin(\theta/2) \\ 0 \\ \cos(\theta/2) \\ \frac{d}{2}\cos(\theta/2) \\ 0 \\ \frac{d}{2}\sin(\theta/2) \end{pmatrix}.$$

This gives parameterisations of the surfaces, so that we can write the coordinates of points on these surfaces as,

$$\begin{aligned} x &= \lambda\cos(\theta/2), & x &= -\mu\sin(\theta/2), \\ y &= d/2 = \frac{p}{2}\tan\theta, \quad \text{or} & y &= d/2 = \frac{p}{2}\tan\theta, \\ z &= \lambda\sin(\theta/2), & z &= \cos(\theta/2), \end{aligned}$$

where λ and μ are arbitrary parameters. In both cases, if we eliminate the parameters, θ and λ or μ, we get the same equation for the ruled surface,

$$y = p\frac{xz}{x^2 - z^2},$$

which can also be written as,

$$y(x^2 - z^2) - pxz = 0.$$

The ruled surface determined by the axes of the half-turns generating the motion is an example of Zindler's conoid, see [14] for example. The surface is illustrated in Fig. 4.

In [15], Röschel finds all rational rigid-body motions that produce generic trajectories that are rational quartic curves. The motion here is an example of a type outlined in chapter 3 of [15] with normal form given by equation (3.7). All the motions considered in this chapter are line symmetric but the particular case where the base surface of the motion is Zindler's conoid was not considered in detail.

2.6. A mechanism to generate the motion

Finally here, we return to the problem of designing a mechanism that will allow a line to move but only take positions in the general line complex. A straightforward solution to this is the Schilling model referred to by Klein [8]. Presumably, the final

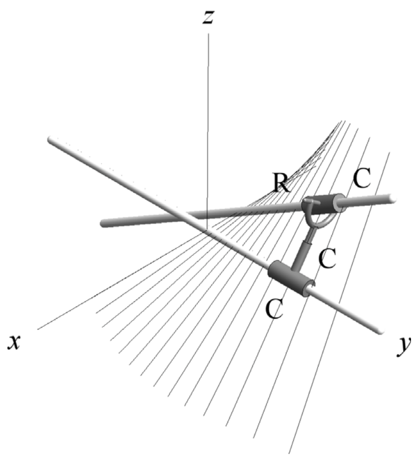

Fig. 5. Closed loop CCRC Mechanism to move a line in a rectangular hyperbolic paraboloid. The grounded links are white and the moving links are coloured grey. Cylindrical joints are labelled C and the revolute joint is labelled R.

cylindrical joint had a groove on the cylindrical part shaped as the quartic curve found above, and a corresponding notch on the mating part so that the end-effector followed the motion given in (2), see Fig. 3.

Here, the aim is to design a simple mechanism consisting of rigid links and lower Reuleaux pairs as joints only. Since the lines in the complex can be arbitrarily far from the axis of the complex we cannot expect to be able to use only revolute joints and we will have to be content with only reproducing part of the complex of lines due to the finite extension of any real joint.

To begin with, restricting a line to be in the orbit of a cylindrical subgroup of $SE(3)$ is straightforward. All that is required is a cylindrical joint which could also be implemented using a revolute joint and a prismatic joint. Where the axis of the revolute joint and the direction of the prismatic joint are parallel. In our case, the axis of the cylindrical joint must meet and be perpendicular to the axis of the line complex. There is no simple, open chain of joints, that will produce the motion given by Eq. (2) — finding a more sophisticated, but still relatively simple linkage is the aim here. There are simple open chain linkages that can move a line so that it remains in a special linear line complex. That is, the set of lines reciprocal to a fixed line. For example, the UPU linkages studied in [16] could be used, as could a PSP chain or the CRC chain. In $\mathbb{P}^7$ the constraint variety for such a geometry has the same form as given above for lines in a general linear complex. Consider the set of displacements that maintain the same initial line $\ell(0)$, but now in the special linear complex of all lines meeting one of the lines in the second regulus $\ell'(\phi)$ for some ϕ. This is a quadric with the equation,

$$\begin{aligned}&\big(p(a_1a_3 - a_0a_2) - (a_0c_2 - a_1c_3 + a_2c_0 - a_3c_1)\big)\sin\phi - \\ &\quad\big(p(a_0a_3 + a_1a_2)\sin\phi\tan\phi + (a_0c_3 + c_0a_3 + a_1c_2 + a_2c_1)\big)\cos\phi = 0.\end{aligned}$$

Now, intersecting this with the 3-plane of cylindrical displacements, $a_1 = a_3 = c_1 = c_3 = 0$ produces, $pa_0a_2 + a_0c_2 + a_2c_0 = 0$. This is the same quadric that we found for the lines in the general linear complex. Moreover, this does not depend on ϕ, so any line in the second regulus could be used as the axis of the special linear complex. The proposed mechanism is illustrated in Fig. 5. In this case a CRC chain has been chosen to provide the quadratic constraint.

There is one final point to check. The intersection of two quadrics in $\mathbb{P}^3$ will give a quartic curve. The discussion above shows that in our case, the intersection of the Study quadric the quadratic constraint variety and the $\mathbb{P}^3$ containing the cylindrical subgroup, will contain a component that is a twisted cubic curve. Hence, there must be another component, necessarily a line. Depending on its location, this line could interfere with the functioning of the mechanism. However, the line is clearly given by $a_0 = a_2 = 0$ and hence lies on A_∞, the 3-plane of ideal or unphysical displacements, so this component will have no influence on any practical realisation of the mechanism.

The CCRC mechanism shown in Fig. 5, only produces a motion that moves a line in a regulus of a rectangular hyperbolic paraboloid. To get the full range of motion such that the line can be placed anywhere in the linear complex all that is needed is to mount the mechanism above on another cylindrical joint, so that the axis of the joint is the same as the axis of the complex. Notice that, the resulting mechanism is not a serial chain nor a single loop mechanism. It would consist of a cylindrical joint in series with a CCRC loop.

3. More simple examples of ruled surfaces and mechanisms

The above considerations prompt questions about other ruled surfaces. In particular, can we design a mechanism to move a line along some other ruled surfaces? The simplest ruled surface that can be considered would be the regulus of a hyperboloid of one sheet. If we require the ruled surface to be a regulus of a circularly symmetric hyperboloid, then the answer is trivial. It is well known that rotating a line about an axis skew to the line will produce such a ruled surface. In other words, the required mechanism is a single revolute joint.

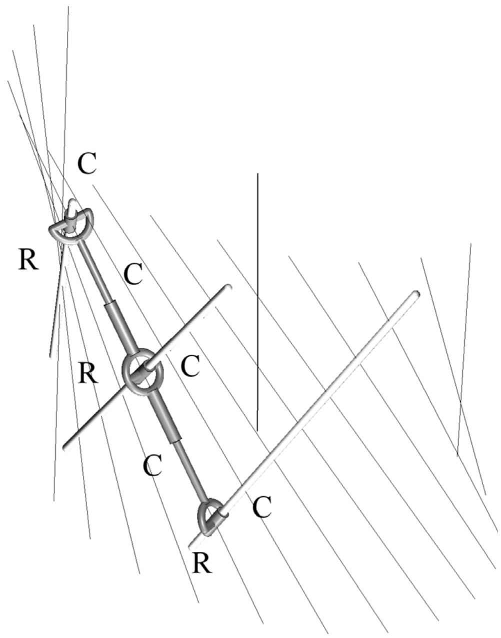

Fig. 6. Mechanism to guide a line along a regulus of an elliptical hyperboloid. Again the fixed links are white and the revolute and cylindrical joints are labelled R and C respectively.

3.1. The regulus of an elliptic hyperboloid

If the required ruled surface to be traced is a regulus in an elliptic hyperboloid then the problem is a little harder. Suppose the surface is given by the equation,

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1. \tag{5}$$

The lines in the surface can be parameterised as,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha\cos\phi - \alpha\lambda\sin\phi \\ \beta\sin\phi + \beta\lambda\cos\phi \\ \pm\gamma\lambda \end{pmatrix},$$

where different values of ϕ give different lines and λ is the parameter along the line. The two reguli are distinguished by the different signs for the last component.

The Plücker coordinates of these two families of lines are given by,

$$\ell^+(\phi) = \begin{pmatrix} -\alpha\sin\phi \\ \beta\cos\phi \\ \gamma \\ \beta\gamma\sin\phi \\ -\alpha\gamma\cos\phi \\ \alpha\beta \end{pmatrix} \quad \text{and} \quad \ell^-(\psi) = \begin{pmatrix} -\alpha\sin\psi \\ \beta\cos\psi \\ -\gamma \\ -\beta\gamma\sin\psi \\ \alpha\gamma\cos\psi \\ \alpha\beta \end{pmatrix}.$$

It is easy to see that $\left(\ell^+(\phi)\right)^T Q_0\left(\ell^-(\psi)\right) = 0$ and hence every line in one regulus meets every line in the other family. This can be used to design the mechanism. This time we need three fixed lines from the other regulus, say from ℓ^-. Then, RC joints allow the line to meet each fixed line but with all possible motion of the line within the special linear line complex defined by the fixed lines. A pair of C joints coaxial with the moving line then allow rotation about and translation along the line, see Fig. 6. Notice, this is not a single loop mechanism.

The fact that the hyperboloid was elliptical in section rather than circular was not important here. So, it should be possible to construct such a mechanism to move a line along the regulus of a circular hyperboloid.

Also note that there is at least one other mechanism that will produce a motion that guides a line along such a regulus. An elliptic trammel is a planar mechanism that can draw an ellipse. That is, a point on the coupler bar of such a mechanism will describe an ellipse. Intersecting the regulus of an elliptic hyperboloid with a plane perpendicular to its axis gives an ellipse. Each point of the ellipse lies on a single generator line of the regulus. However, if we rigidly attach a line to the point on the coupler of an elliptic trammel, the angle between the line and the line through the centre of the ellipse perpendicular to the plane of the mechanism will not be constant. So, a slightly more complex mechanism is required to solve this problem.

3.2. The cylindroid

The cylindroid or Plücker's conoid is a ubiquitous figure in kinematics and the theory of mechanisms. It is a cubic surface with the canonical equation,

$$(x^2 + y^2)z = 2xy.$$

As a ruled surface, the generator lines can be parameterised as,

$$\ell = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ \sin\theta\sin 2\theta \\ \cos\theta\sin 2\theta \\ 0 \end{pmatrix} = \begin{pmatrix} c(c^2+s^2) \\ s(c^2+s^2) \\ 0 \\ 2cs^2 \\ 2c^2s \\ 0 \end{pmatrix},$$

where $c = \cos\theta$ and $s = \sin\theta$ are homogeneous parameters. This shows that the ruled surface is represented as a twisted cubic curve in the Klein quadric of lines in $\mathbb{P}^3$.

The fact that the surface is a conoid, that is, all the lines meet a single fixed line at right-angles, means that we can find a simple parameterised motion that will move one line in the surface to each of the other lines comprising the surface. The desired motion is given by translating along the axis of the conoid and rotating around this axis,

$$\begin{aligned} G(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sin 2\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \sin 2\theta \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

In the Study quadric this motion is represented by a rational curve of degree 5,

$$g(\theta) = \big(c(c^2+s^2)^2 + s(c^2+s^2)^2 k\big) + \varepsilon\big(-2cs^2(c^2-s^2) + 2c^2s(c^2-s^2)k\big), \tag{6}$$

where here $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$. The intersection with A_∞ is given by the complex points $c = \pm is$ repeated twice. Applying this motion to a point in space produces a rational quartic curve lying on a cylinder with the same axis as the conoid,

$$\begin{aligned} X &= x(c^2-s^2)(c^2+s^2) - 2ycs(c^2+s^2), \\ Y &= 2xcs(c^2+s^2) + y(c^2-s^2)(c^2+s^2), \\ Z &= z(c^2+s^2)^2 + 4cs(c^2-s^2), \\ W &= w(c^2+s^2)^2. \end{aligned}$$

Again the parameters represent the sine and cosine of the half-angles here.

Consider a regulus of a hyperboloid. Assume that the hyperboloid is given by (5) as in the previous section. Further, assume that the regulus is given by $\ell^+(\phi)$ as in the previous section again. Now consider the ruled surface formed from the common perpendicular lines between one line in the regulus, $\ell^+(0)$ say, and all the other lines in the regulus, $\ell^+(\phi)$ for all ϕ. Since $\ell^+(0)$ is constant all the lines in the ruled surface will meet $\ell^+(0)$ perpendicularly, so the surface is a conoid. The common perpendicular to the lines is given by the axis of the screw product,

$$\ell^+(0) \times \ell^+(\phi) = \begin{pmatrix} \beta\gamma(1-\cos\phi) \\ -\alpha\gamma\sin\phi \\ \alpha\beta\sin\phi \\ \alpha(\beta^2-\gamma^2)(1-\cos\phi) \\ \beta(\gamma^2-\alpha^2)\sin\phi \\ -\gamma(\alpha^2+\beta^2)\sin\phi \end{pmatrix}$$

If we use half-angles to parameterise this twist we get a common factor of $2\sin(\phi/2)$ which can be cancelled since these are homogeneous Plücker coordinates. The result can be written as,

$$\ell^+(0) \times \ell^+(\phi) = s\begin{pmatrix} \beta\gamma \\ 0 \\ 0 \\ \alpha(\beta^2-\gamma^2) \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -\alpha\gamma \\ \alpha\beta \\ 0 \\ \beta(\gamma^2-\alpha^2) \\ -\gamma(\alpha^2+\beta^2) \end{pmatrix},$$

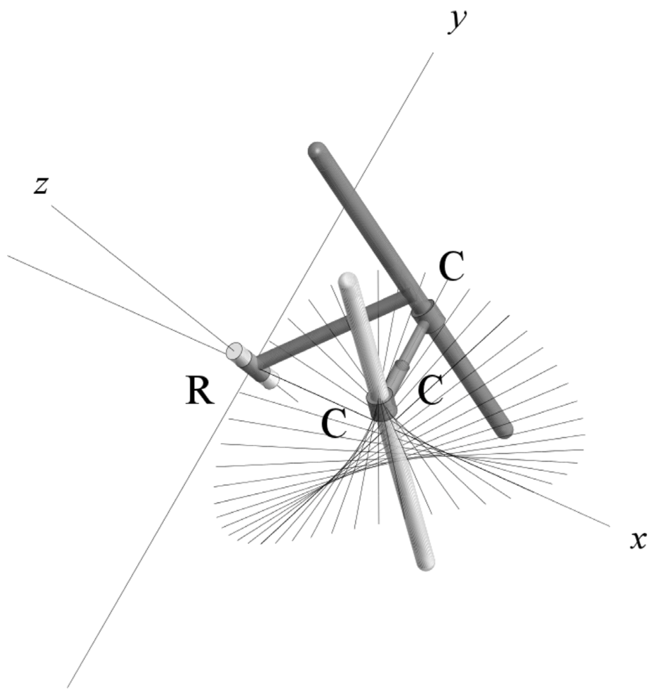

Fig. 7. Mechanism to guide a line along a cylindroid. Again the fixed links are white and the revolute and cylindrical joints are labelled R and C respectively.

where $s = \sin(\phi/2)$ and $c = \cos(\phi/2)$. This is a linear 2-system of screws with homogeneous parameters s and c, it is well known that the axes of the screws in such a system form a cylindroid, see [10] for example.

Notice, this is really a theorem in classical geometry: The common perpendiculars from one line in a regulus to every other line in a regulus form a cylindroid. However, the authors were not able to find this in the classical literature. The principal pitches of the 2-system can be found to be p_a, $p_b = -\beta\gamma/\alpha$, $\alpha(\beta^2 - \gamma^2)/(\beta\gamma)$.

From the above result it is straightforward to design a mechanism to guide a line along a cylindroid. The simplest case is when the regulus belongs to a circularly symmetric hyperboloid, that is when $\alpha = \beta$. We can place a revolute joint along the axis of the hyperboloid and fix a cylindrical joint along one of the lines in the regulus. The revolute joint swings the axis of another cylindrical joint through the other lines of the regulus. Finally, a third cylindrical joint connects the other two cylindrical joints. The axis of this third cylindrical joint is perpendicular to the axes of the other two cylindrical joints and hence describes a cylindroid as the mechanism moves, see Fig. 7. The single loop mechanism shown in Fig. 7 does not take account of mechanical interference. It is intended to illustrate the concept only.

Finally here, we mention that an alternative mechanism could be found for this task by factorising the degree 5 motion polynomial found in Eq. (6) using the methods given in [17]. Alternatively, there might be a mechanism based on the Bennett mechanism using the associated cylindroid found by Huang [18].

4. Two and three parameter examples

4.1. Special linear line complexes

The, so called, special linear line complex corresponds to a pitch zero screw. Hence it consists of all lines reciprocal to a fixed line. That is, all lines coplanar to a fixed line. In the Study quadric the group elements that keep a line in a special linear congruence satisfy a quadratic equation. The equation is the same as the equation for the general linear complex given in (4), only the blocks Ξ and Υ reflect the fact that the fixed screw, $\mathbf{s}^T = (\mathbf{w}^T, \mathbf{u}^T)$ is, in fact, a line.

There are several open loop mechanisms that can move a line in such a way that it remains in a special linear line complex, for example the PSP or UPU linkages, see [16].

Line complexes which correspond to infinite pitch screws do not seem to have a particular name. It is straightforward to see that the set of lines reciprocal to an infinite screw comprise the set of lines in space perpendicular to a fixed direction. That is, the lines lying in a set of parallel planes. The equation for the group elements in the Study quadric which maintain a line in such a complex is again given by a quadratic. This time, however, the diagonal block Υ, of the symmetric matrix in (4), is the 3×3 zero matrix, since $\mathbf{w} = \mathbf{0}$. This represents a singular quadric in $\mathbb{P}^7$, where the singular set of the quadric is clearly the 3-plane A_∞.

It is not difficult to find a linkage that will move a line so that it remains in such a complex. For example, a CPC mechanism where the two C-joints are perpendicular and the middle P-joint is directed perpendicular to both cylindrical joints will do the job.

This prompts consideration of a closely related set of lines. Consider the set of lines in space that make a constant angle with a fixed line. The Plücker coordinates of such lines $\ell^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$, might be considered as satisfying an affine equation,

$$\mathbf{u} \cdot \boldsymbol{\omega} = \cos\alpha,$$

for a fixed angle α. In $\mathbb{P}^5$ affine equations are not valid, but we can find a homogeneous equation for these lines by assuming that $\boldsymbol{\omega} \cdot \boldsymbol{\omega} \neq 0$. Then we can write a quadratic equation for the Plücker coordinates of the lines,

$$(\mathbf{u} \cdot \boldsymbol{\omega})^2 = \cos^2\alpha(\boldsymbol{\omega} \cdot \boldsymbol{\omega}).$$

A set of lines determined by a quadratic equation in the Plücker coordinates is called a quadratic line complex. These were objects of much classical study, see for example, [19]. The quadratic equation above corresponds to a singular quadric in $\mathbb{P}^5$, where the singular set of the variety is the set of lines at infinity $\boldsymbol{\omega} = \mathbf{0}$.

It is not difficult to see that a CPC linkage will keep the axis of the final C-joint in the finite part of this quadratic line complex if the first and final C-joint are set at an angle of α and the P-joint is directed parallel to the common perpendicular between the axes of the two C-joints.

4.2. Linear line congruences

A line congruence is a two-parameter family of lines. A linear line congruence is the intersection of the Klein quadric of lines with a 3-dimensional plane in $\mathbb{P}^5$. Put another way, a linear line congruence is the intersection of a pair of line complexes. Suppose that the two line complexes are determined by a pair of screws $\mathbf{s}_1$ and $\mathbf{s}_2$, then the congruence will also be contained in complexes determined by any linear combination of the screws, $\lambda\mathbf{s}_1 + \mu\mathbf{s}_2$. That is, for any constants λ and μ the congruence will satisfy,

$$(\lambda\mathbf{s}_1 + \mu\mathbf{s}_2)^T Q_0 \ell = 0.$$

This means that a congruence is determined by a 2-system of screws. Traditionally, the classification of these congruences has considered different congruences to be equivalent if they are related by a real projective transformation (projectivity), see [14]. Three types of linear line congruences are considered, these are distinguished by the number of real or imaginary lines in the 2-system of screws, that is the character of the solutions to the quadratic,

$$\begin{aligned} 0 &= (\lambda\mathbf{s}_1 + \mu\mathbf{s}_2)^T Q_0 (\lambda\mathbf{s}_1 + \mu\mathbf{s}_2) \\ &= \lambda^2\left(\mathbf{s}_1^T Q_0 \mathbf{s}_1\right) + 2\lambda\mu\left(\mathbf{s}_1^T Q_0 \mathbf{s}_2\right) + \mu\left(\mathbf{s}_2^T Q_0 \mathbf{s}_2\right). \end{aligned} \tag{7}$$

If both solutions to this quadratic are complex then the congruence is said to be an elliptic congruence, when the quadratic has two real roots the congruence is called hyperbolic and the congruence is referred to a being parabolic if the root is repeated.

As mentioned, linear line congruences are determined by 2-systems of screws. So, a finer classification of these objects exists up to equivalence by rigid-body transformations, see [10]. The elliptic, hyperbolic and parabolic congruences are all examples of lines reciprocal to a IA 2-system (the general 2-system in Hunt's classification). The canonical form for the IA 2-system is given by basis elements,

$$\mathbf{s}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ p_a \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ p_b \\ 0 \end{pmatrix}.$$

Here, p_a and p_b are know as the principal screws of the system. The quadratic which determines the pitch zero screws in the system and hence the character of the congruence is,

$$\lambda^2 p_a + \mu^2 p_b = 0.$$

Hence, the congruence will be elliptic if the principal pitches have the same sign, $p_a p_b > 0$. It will be hyperbolic if they have opposite signs $p_a p_b < 0$. The congruence will be parabolic if $p_b = -p_a$.

An elliptic congruence consists of the lines from one regulus of a system of nested cylindrical hyperboloids, see [14]. This is illustrated in Fig. 8. From the above, it is simple to give a 2 degree-of-freedom mechanism that can move a line so that it remains in such a congruence. Clearly, the lines in this congruence are the restriction of the lines in a single linear complex. So, if we mount the mechanism shown in Fig. 5 on an R-joint whose axis is aligned with axis of the set of hyperboloids, then line determined by the middle C-joint of the mechanism will trace out the lines in the elliptic congruence.

When the quadratic in (7) has two real roots the 2-system of screws contains two real lines and all the lines of the congruence will be reciprocal to these two lines. That is, the hyperbolic linear line congruence consists of all lines that meet a pair of lines. A mechanism can be readily designed that will move a line in such a congruence. All that is needed is to join a pair of linkages that maintain the incidence between a moving line and a fixed one. In Section 2.6, a CRC linkage was used for this. To make the point that different choices are possible, the mechanism shown in Fig. 9 uses a pair of PSP linkages. The two P-joints where the

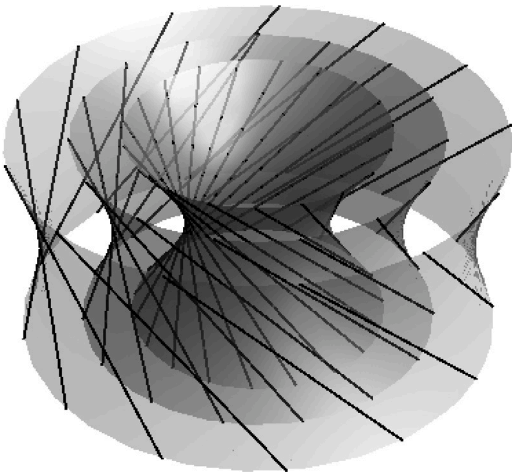

Fig. 8. The elliptic linear line congruence.

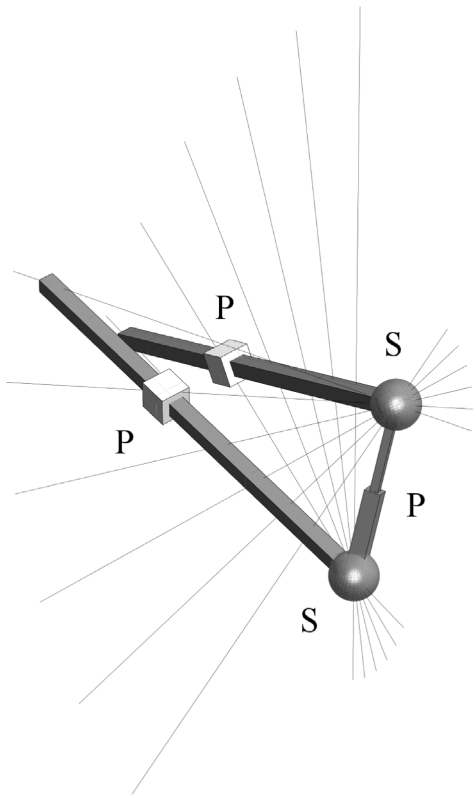

Fig. 9. Closed loop PSPSP mechanism to move a line in a hyperbolic linear line congruence. The prismatic joints are labelled P and the spherical joint are labelled S. The white links are the fixed links.

linkages are joined can be replaced by a single P-joint to create a 2 degree-of-freedom mechanism that moves a line in space so that it remains in a hyperbolic linear line congruence.

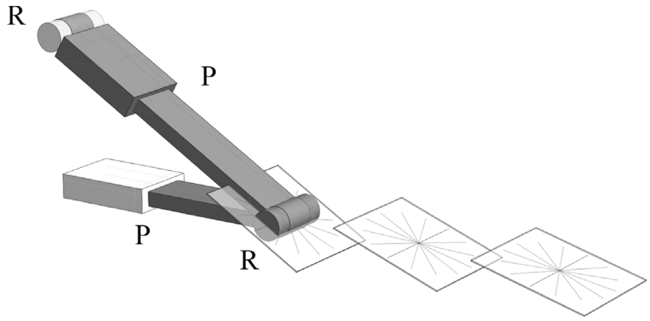

Fig. 10. An RPRP mechanism to move a line in a parabolic linear line congruence. The lines in the congruence will lie in the plane attached to the upper limb of the triangle and pass through the centre of the lower R-joint. Again, the fixed links are coloured white.

Turning to the parabolic linear line congruence, we can use the canonical basis given above but with $p_a = p$ and hence $p_b = -p$. The single line in the screw system is given by,

$$\mathbf{s}_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ p \\ -p \\ 0 \end{pmatrix},$$

points on this line are given by,

$$\mathbf{r} = \begin{pmatrix} v \\ v \\ -p \end{pmatrix},$$

where v is an arbitrary parameter. So lines in the singular complex determined by this line can be written as,

$$\ell = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{r} \times \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \omega_y p + \omega_z v \\ -\omega_x p - \omega_z v \\ -\omega_x v + \omega_y v \end{pmatrix}.$$

Setting the reciprocal product of these lines with $\mathbf{s}_1$ to zero gives the lines in the congruence,

$$\mathbf{s}_2^T Q_0 \ell = \omega_x p + \omega_y p + \omega_z v = \left(p,\ p,\ v\right) \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0.$$

The lines in this parabolic congruence consist of lines meeting ℓ but through each point on ℓ we get a plane star of lines. If the point is determined by the parameter v then the normal vector to the plane is $\left(p,\ p,\ v\right)^T$. When $v = 0$, that is the point on the line closest to the origin, the plane of lines is normal to the line ℓ. To produce a mechanism to move a line such that it is constrained to a parabolic congruence consider the planar triangular RPRP mechanism shown in Fig. 10. As the lower limb of the triangle extends, simple trigonometry shows that a plane attached to the upper limb will follow the motion of the plane of lines in the congruence. This plane has to be centred at the lower R-joint so that it follows the straight line determined by the lower limb. This design would make it difficult to include an R-joint to rotate the line in the plane due to mechanical interference.

5. Conclusions

The mechanisms proposed above may not have many immediately obvious uses. The purpose of this work has been to give some examples of simple mechanisms that address the problem of guiding a line through a predetermined set of lines. However, the shape of the quartic curve traced by points on the line in the regulus suggests that there might be applications as a bistable or switching mechanism.

This design method illustrated in this work relies on an extensive knowledge of the geometry of the problem. The idea is to have a sufficient knowledge of the geometry of the curve describing the motion, in particular the subvarieties that it can lie in. We also require a knowledge of the constraint varieties, or displacement varieties, that are generated by a range of different open loop chains.

Finally, as mentioned above, there is a classification of 2-systems of screws due to Gibson and Hunt, [20,21]. A closer study of the congruences of lines reciprocal to these system can easily be envisaged.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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