Some Rigid-Body Constraint Varieties Generated by Linkages

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Introduction

Rather than go through paper in detail, I want to give a broard overview of the work and how it relates to several familar ideas in kinematics.

(I'l explain what I mean by Constraint Variety after a few slides).

Inspiration I

Current work tries to generalise two ideas. First of these was work by Jacques Hervé.

Consider two mechanisms which generate (codimension 0 sets of) motions in different Schoenflies subgroups of $SE(3)$.

Joining such mechanisms in parallel the possible displacements that the platform can undergo is given by the intersection of the two Schoenflies groups. This is, in general, the group of translation $\mathbb{R}^3$, or some subset of this subgroup.

Notice, almost no calculation needed to produce this result.

Inspiration II

The second idea that I want to generalise is the work of Ian Parkin, Ken Hunt, Chintien Huang and several others, on finite screw systems.

If rigid body displacements are represented by dual quaternions, then the set of all such displacements represented by a 6-dimensional quadric in $\mathbb{P}^7$—the Study quadric. (Actually an open set in this variety, need to delete one A-plane). A group element is represented by a dual quaternion of the form,

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k)$$

Inspiration II — Finite Screw Systems

To be a group element the dual quaternions must satisfy the condition,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$$

the equation of the Study quadric. Here i, j, k are usual quaternion basis and ε is the dual unit which squares to zero $\varepsilon^2 = 0$ and commutes with the quaternions.

A finite screw system is the set of displacements lying on a linear subspace of $\mathbb{P}^7$. This might lie inside the Study quadric or intersect it.

Some original skepticism about this idea. Why only $\mathbb{P}^7$?

Constraint Varieties

A **constraint variety** is any set of rigid body displacements defined as an algebraic subvariety of the Study quadric. (Variety=Algebraic set).

Poor name, originally thought of varieties of displacements defined by geometric constraints. For example work by Charles Wampler on sets of displacements which maintain the contact between a point on the rigid body and a fixed plane.

I would call a finite screw system a linear constraint variety.

Linear Constraint varieties

Many well known examples. Just look at a few in the next couple of slides.

The Study quadric contains 2 families of 3-planes usually called, A-planes and B-planes. One of these A-planes is distinguished, its points are not physical group elements — called the A-plane at infinity, denoted A_∞ and given by $a_0 = a_1 = a_2 = a_3 = 0$.

A-planes through the identity ($g = 1$), come in 2 types:

- those that don't meet A_∞ are $SO(3)$ subgroups, *i.e.* rotations about some point in space.
- those that meet A_∞ in a line are $SE(3)$ subgroups, *i.e.* displacements preserving some plane in space.

Cylindrical Subgroups

The set of rotations about some axis combined with translations in the direction of the axis form a subgroup of $SE(3)$. This subgroup can be thought of as the displacements generated by a cylindrical joint.

The set of group elements that are generated by a C joint form a Segre variety. It is simple to see that this variety can be parameterised by a rotation about the axis followed by a translation along the axis so the variety generated is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

In algebraic geometry this is well known to be isomorpic to a 2-dimensional quadric. Hence the variety must be the intersection of the Study quadric with some 3-plane (not lying in the Study quadric).

A little further work reveals that this 3-plane must meet A_∞ in line.

Two 4D Linear Constraints

- The Schoenflies subgroups mentioned earlier, all lie on the intersection of the Study quadric with a 5-plane. Such a 5-plane will contain A_∞.
- Any line-symmetric motion lies in a 5-plane. First of all, a rigid-body motion can be thought of as a curve in the Study quadric. The 5-plane that the curve lies in will depend on the lines in the ruled surface generating the motion. However, such a 5-plane will meet A_∞ in a 2-plane.

Geometrie der Dynamen

Eduard Study called these linear constraint varieties: chains ("Kette").

He claimed to have investigated them systematically and given a kinematic interpretation for each in his 1903 book.

Res severa verum gaudium

Examples - What is this used for?

There many ways to use the information on constraints varieties, look at some examples in the next few slides.

Suppose we connect in parallel a cylindical joint and a mechanism which constrains a point on the coupler to lie in a plane. What motion do we expect from the coupler?

Now it is possible to show that the constraint variety generated a point-plane constraint is the intersection of the Study quadric with another quadric in $\mathbb{P}^7$. This point-plane quadric also contains A_∞.

Hence we seek the intersection of two quadrics with a 3-plane. In general this would be a curve of degree $2 \times 2 = 4$. However, we have seen above that the 3-plane determined by the cylindric joint intersects A_∞ in a line, so this line must be a component of the intersection of all three varieties. The intersection is therefore a twisted cubic curve.

Examining the singularities of the intersection reveals that the singularities lie on an imaginary 2-sphere in A_∞. This 2-sphere is invariant under the action of $SE(3)$, I call it S_∞. The intersection consists of a twisted cubic curve and a line which meets the curve in a pair of points on S_∞.

The famous Darboux motion can be charaterised as a twisted cubic curve in the Study quadric which meets S_∞ at two points.

N.B. this result was know to Study.

Another Example

Husty *et al* showed that the displacements allowed by an RRR linkage is the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This variety is the intersection of 9 quadrics in $\mathbb{P}^7$ one of which can be taken as the Study quadric.

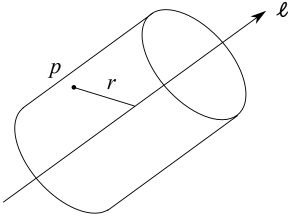
Suppose we join two RRRs, how many possible ways are there to do this? Notice this question is equivalent to counting the number of postures of for a 6R manipulator.

The degree of this Segre variety is 6. This is not enough to compute the number of intersections. Three dimensional subvarieties of the Study quadric have a bi-degree given by the number of intersection with a generic A-plane and the number of intersections with a generic B-plane. For the RRR constraint variety this bi-degree is (4, 2). The rules for combining these bi-degrees give,

$$(4, 2) \cap (4, 2) = 4 \times 2 + 2 \times 4 = 16$$

postures for the 6R robot.

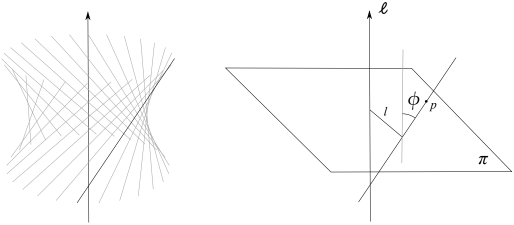
The CS Linkage



In the paper look at 3 examples, these come from Mike McCarthy and Gim Song Soh's book. Here, look at the examples in $\mathbb{P}^7$.

- Displacements generated by a CS linkage same as displacements required to keep point (centre of S joint) on a cylinder.
- Constraint variety is intersection of Study quadric with a quartic hypersurface.
- This quartic contains A_∞, moreover the variety is singular along A_∞.

The RPS Linkage

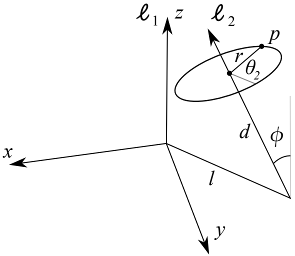


(a) (b)

Group elements which maintain contact between a point and a cylindrical hyperboloid.

Again constraint variety is a quartic hypersurface (intersecting the Study quadric) and again it contains and is singular along A_∞.

The RRS Linkage



Sweeping circle about axis gives 'generalised' torus.
Now constraint variety is a degree 8 hypersurface (intersecting the Study quadric) but again it contains and is singular along A_∞.

Conclusions

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- Can't handle helical joints — not algebraic.
- Do need to consider other models of $SE(3)$.
- Also think about space of all CS linkages, for example, for synthesis problems.