# Finite size effects in the averaged eigenvalue density of Wigner random-sign real symmetric matrices

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Nowadays, strict finite size effects must be taken into account in condensed matter problems when treated through models based on lattices or graphs. On the other hand, the cases of directed bonds or links are known to be highly relevant in topics ranging from ferroelectrics to quotation networks. Combining these two points leads us to examine finite size random matrices. To obtain basic materials properties, the Green's function associated with the matrix has to be calculated. To obtain the first finite size correction, a perturbative scheme is hereby developed within the framework of the replica method. The averaged eigenvalue spectrum and the corresponding Green's function of Wigner random sign real symmetric $N \times N$ matrices to order $1/N$ are finally obtained analytically. Related simulation results are also presented. The agreement is excellent between the analytical formulas and finite size matrix numerical diagonalization results, confirming the correctness of the first-order finite size expression.

## I. INTRODUCTION

Random matrix studies can be traced back a long time but are an intense research subject nowadays [1]. They were an essential part of discoveries in nuclear physics [2,3]. Thereafter, the emergence of amorphous and/or disordered alloys as interesting materials led to the consideration of the distribution of eigenvalues $\lambda$ for the Green's function (itself a random matrix) associated with the Hamiltonian describing the system [4]; see Eq. (10) below.

In brief, it is well known that the density of states is the imaginary (Im) part of this Green's function, the energy states being the eigenvalues [5,6]. Thus, the eigenvalue spectrum has to be well known, in particular, to determine the presence of (optical or conduction) spectral gaps and state localization, such as in the context of the Anderson model [7,8] and spin glass models [9]. Moreover, the largest (necessarily real according to the Perron–Frobenius theorem) and the next-to-largest eigenvalues (not necessarily real in the case of asymmetric matrices) are, for the former, indications of the ground state of the system, through an approximation of the free energy and of the diffusion (or relaxation) coefficient for the latter. From a "more general" point of view, let us simply say that the averaged eigenvalue density (AED) and its properties have to be calculated, or must receive some theoretical estimate with enough precision taking into account the system finite size—the best being when searching for universal features.

There is a large body of mathematical work on the spectra of random matrices, ranging from modern versions of Perron–Frobenius theorem for non-negative matrices [10,11], up to recent results reviewed in Refs. [12–17]. In fact, a paper on "finite size corrections to disordered Ising models," on random regular graphs [18] recently appeared [19], indicating the up-to-date interest of considering binary distributions

in the context of random matrices and graphs. When this paper was in its final stage, a review [20] appeared on the "random-matrix theory of Majorana fermions and topological superconductors," indicating the up-to-date interest in the matter.

One of the most studied ensembles of random matrices is the Gaussian orthogonal ensemble (GOE) of $N \times N$ real symmetric matrices, which is invariant under orthogonal transformations [21–24]. Much attention has focused on calculating its AED. It was shown by Wigner [25] that the AED for the GOE in the limiting case of $N$ going to infinity is a semicircle. This, among many other early results in the field of random matrices, can be found in the early texts by Porter [26] and by Mehta [27]. Most of the investigation methods rely either on elaborate moment and cumulant expansions or on the properties of orthogonal polynomials [28–30]. A radically different method was presented by Edwards and Jones [31] for calculating the AED. The Edwards and Jones (EJ) method relied on the so called "replica trick" first employed by Edwards [32] in the study of polymer physics, reviewed by Advani *et al.* [33], for example, which led to the renormalization group technique later on.

By no means has all the published work on random matrices been directed at the GOE. Wigner [34] addressed the problem of calculating the AED of an ensemble of large symmetric random matrices that were either bordered, i.e., having integers along the diagonal and random numbers equal to plus or minus some constant $J$ on the super- and sub-diagonals, or had zeros on the diagonal and entries that (subject to the symmetry requirement) were either $+J$ or $-J$ in the off-diagonal elements. For example, let us have in mind an Ising spin system in which two neighbors are pointing in different or in similar directions, thus resulting in a $-J$ or $+J$ bonding energy; dipoles in ferroelectric materials can also be considered as being at different energy levels anti-symmetrically placed with respect to the zero level. Another case is the fully connected network where links can take two different weights—here they should be equal in magnitude but with opposite signs. The latter of these two ensembles

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is best called the random sign symmetric matrix ensemble (RSSME).

In a short article, Wigner [25] conjectured that a semicircular distribution of eigenvalues would be the limiting distribution obtained as $N \to \infty$ for an ensemble of symmetric matrices in which the probability density function (pdf) of any off-diagonal element is reasonably well behaved and for which the second moment of all such off-diagonal elements should have the same constant value. It is worth pointing out that this is akin to the AED of a $d$-regular random graph with $N$ vertices for $N \to \infty$ and $d \to \infty$. This is due to the tree-like structures that emerge when calculating the AED of these sets of random matrices. Hence, the AED becomes a semicircle as in the GOE of matrices, in the limit of size $N \to \infty$, and diverging mean $d \to \infty$, i.e., when the Kesten–Mckay law converges to a semicircular law [35–40]

However, no real system has an infinite size. The more so in "real world" and "subsequent" applications. The finite size constraint must be nowadays taken at its full value, even though it was previously often taken as an irrelevant nonuniversal effect in many condensed matter investigations. Very often, the surface energy and surface entropy terms were disregarded in free energy calculations.

Jones and Dhesi [41] (hereafter referred to as JD) applied the replica method to the case of the RSSME and showed, in the limit of $N \to \infty$, how easily the replica formalism produces a semicircular AED. (By using the same formalism, they were able to verify the Wigner conjecture.) This leads to the interesting problem of calculating the AED when $N$ is finite. In such a case ($N$ finite), the AED distribution departs from the semicircle and becomes ensemble specific. For the GOE, there has been a number of papers that have addressed the problem of calculating the corrections to the Wigner semicircle which are of order $1/N$ [42–44]. Dhesi and Jones [45] (thereafter referred to as DJ) have provided a comprehensive set of results: DJ calculated the AED for the GOE to order $1/N^2$, and also the AED for finite $N$ based on a self-consistency argument, when each element of the matrix is drawn randomly from a normal distribution. Note that these four papers rely on different methods; furthermore, the $1/N$ correction of Takano and Takano [42] differs from the others.

Recently, Metz *et al.* [46] reconsidered finite size corrections to the spectrum of regular random graphs obtaining an analytical solution, given by a sum over loops comprising all length scales, each loop contributing with a term proportional to the difference of its effective resolvent with respect to the resolvent of an infinite closed chain. In this context, let us also point to Kanzieper and Akemann [47] who looked "through the prism of probabilities" on how to find exactly a given number of *real* eigenvalues in the spectrum of an $N \times N$ real asymmetric Gaussian random matrix (see some elaboration in the conclusion). Finally, to obtain an overview of relevant applications and subsequent approaches to complex systems, as those of concern here, see Ref. [48].

The present paper is still devoted to the calculation of the AED for the RSSME, with vanishing diagonal elements, although it will be shown that this constraint is rather irrelevant, to order $1/N$, but on an apparently more relevant set of cases, like fully connected random graphs, with a given distribution of different types of links; see below. Nevertheless, to provide
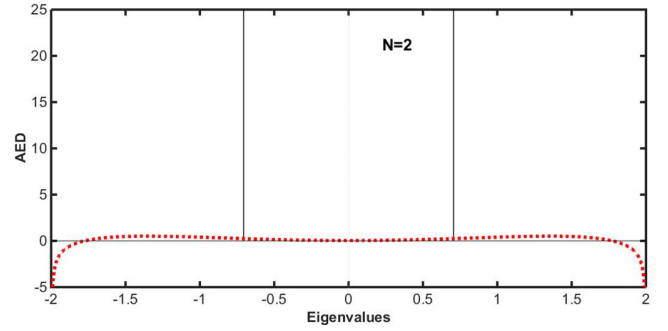


FIG. 1. Comparison of the $N = 2$ "numerical simulation" AED (vertical lines), leading to two delta functions, with the theoretical first-order $O(1/N)$ approximation (red) dotted line, resulting from Eqs. (50) and (51).

a flavor of the problem generality, let us compare the AED for the GOE and the RSSME in both extreme cases, i.e., for $N = 2$ or $N \to \infty$. When $N \to \infty$, the AEDs for both the GOE and the RSSME obey the semicircular distribution. However, when $N = 2$, the AED for the GOE has the following form [49]:

$$\rho_{N=2}(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2} \left[ e^{-\lambda^2} + \sqrt{\pi}\lambda \,\mathrm{erf}(\lambda) \right], \qquad (1)$$

where erf is the error function [50].

For our ensemble, when the diagonal elements are zero, the AED for the $N = 2$ RSSME is merely given by two symmetric delta functions, as per Eq. (11), see below in fact; see Fig. 1 for illustration. Whence it can be noted that the departure from a semicircle, when $N$ is small, is much more acute in the case of the RSSME. Therefore, the analytic result for the AED and the RSSME should describe the broadening and the overlapping of the peaked functions, thereby approaching the semicircular function as $N$ becomes large.

As can be rather easily appreciated, this RSSME, i.e., when an element is drawn randomly with equal probability of being positive or negative (head or tail), is a more difficult exercise than when calculating the case of the GOE, i.e., when a normal type distribution [described by Eq. (1) as treated in DJ [45]] converges to the semicircle as $N$ becomes large. We show that additional terms appear when the finite size is taken into account.

Therefore, the plan of this paper is the following: In Sec. II, we recall the basic replica technique for calculating the AED, and its corresponding Green's function, but geared toward the case of random sign symmetric matrix ensembles. Section III is devoted to casting the AED to order $1/N$ of the RSSME as a zero-dimensional path integral. In Sec. IV, we set up a perturbation theory by using the Hubbard–Stratonovich transformation (auxiliary field identity) and Feynman diagrams, which allows us to calculate the correction to order $1/N$, in Sec. V with the steepest descent method. The correction is found to be nonvanishing and convergent only inside the Wigner semicircle and away from the band edges. Analytic and simulation works are presented. The comparison between the analytical formulas and finite size matrix numerical diagonalization results exhibits an excellent agreement, confirming the correctness of the first-order finite size expression. A few comments are made in Sec. VI.

Numerical simulations are found in Sec. VII; their average is graphically compared to the theoretical expressions.

Finally, in Sec. VIII, we summarize the results and suggest some direction for further work, in view of possibly obtaining related results for more complicated cases and applications.

## II. REPLICA TECHNIQUE

The theoretical development is based on the replica technique [32], which is briefly recalled for completeness within the present framework. Consider a real symmetric $N \times N$ matrix $J$ with eigenvalues $J_i$. The density $v(\lambda)$ of such eigenvalues is given by

$$v(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - J_i), \tag{2}$$

where $v(\lambda)$ has been chosen to be normalized to unity. For a real symmetric matrix, it can be recalled that

$$\det(\Lambda - J) = \prod_i (\lambda - J_i), \tag{3}$$

where $\Lambda$ is the diagonal matrix with element $\lambda$. In the complex plane, giving an infinitesimal imaginary part to $\lambda \to \lambda - i\epsilon$, one has

$$v(\lambda) = \frac{1}{N\pi} \mathrm{Im} \frac{\partial}{\partial \lambda} \ln \det(\Lambda - J). \tag{4}$$

The replica trick, further developed by Edwards and Jones [31] in the context of random graphs, uses

$$\ln(x) = \lim_{n \to 0} \left[ \frac{x^n - 1}{n} \right], \tag{5}$$

so that Eq. (4) reads

$$v(\lambda) = \frac{-2}{N\pi} \mathrm{Im} \frac{\partial}{\partial \lambda} \lim_{n \to 0} \frac{1}{n} [\det^{-1/2}(\Lambda - J)^n - 1]. \tag{6}$$

The determinant (det) can be parametrized as a multiple Fresnel integral [50,51]

$$\det^{-1/2}(\Lambda - J) = \left( \frac{e^{i\pi/4}}{\pi^{1/2}} \right)^N \int_{-\infty}^{\infty} \prod_i dx_i \, e^{-i \sum_{i,j} x_i (\Lambda - J)_{i,j} x_j}. \tag{7}$$

Substituting Eq. (7) into Eq. (6), and assuming that this latter result holds for integer $n$ values and can be continued for $n = 0$, one obtains the fundamental result

$$v(\lambda) = \frac{-2}{N\pi} \mathrm{Im} \frac{\partial}{\partial \lambda} \lim_{n \to 0} \frac{1}{n} \left\{ \left( \frac{e^{i\pi/4}}{\pi^{1/2}} \right)^{Nn} \int_{-\infty}^{\infty} \prod_{i;\alpha} dx_{i;\alpha} \right.$$
$$\left. \times \left\{ \exp \left[ -i \Sigma_{i,j;\alpha} x_i^\alpha (\Lambda - J)_{i,j} x_j^\alpha \right] - 1 \right\} \right\}. \tag{8}$$

The integration is now over the $Nn$ variables $x_i^\alpha$ with $i \in (1, N)$ and $\alpha \in (1, n)$, respectively; the $\lim_{n \to 0}$ being taken at the end of the calculation. Therefore, the AED $\rho(\lambda)$ of an ensemble of real symmetric matrices that has a given pdf $p(J_{i,j})$ is

$$\rho(\lambda) \equiv \langle v(\lambda) \rangle = \int v(\lambda; J_{i,j}) \prod_{i,j} p(J_{i,j}) dJ_{i,j}, \tag{9}$$

where the brackets $\langle \rangle$ imply ensemble averaging.

Recall at this stage that the eigenvalue density is related to the Green's function $\mathcal{G}(\lambda) \equiv (\Lambda - J)^{-1}$ through

$$v(\lambda) = \frac{1}{\pi} \mathrm{Im} \frac{1}{N} \mathrm{Tr} \mathcal{G}(\lambda - i\epsilon), \tag{10}$$

where Tr stands for the trace and $\epsilon$ is supposed to be taken as small and positive; let us call $G(\lambda)$ the average Green's function. Whence the AED $\langle v(\lambda) \rangle \equiv (1/\pi) \mathrm{Im}\, G(\lambda)$ is immediately obtained from Eq. (10) through ensemble averaging.

## III. RANDOM SIGN SYMMETRIC MATRIX ENSEMBLE

Consider the specific case in which a real symmetric matrix (thus imposing $J_{i,j} = J_{j,i}$) has zero on its diagonal ($J_{i,i} = 0$), but the off-diagonal elements $J_{i,j}$ randomly take the value $+J/\sqrt{N}$ or $-J/\sqrt{N}$ with equal probability 0.5. Practically, the sign can indicate whether a bond or link is directed or not and pertains to a ferromagnetic or antiferromagnetic set of spins, or has a given color; for example, the equal probability constraint and the matrix symmetry will be suggested, in Sec. VIII, to be removed in further work. Let $J$ be of the order of unity. The ensemble pdf is described by

$$p(J_{i,j}) = \tfrac{1}{2} \{ \delta(J_{i,j} - J\sqrt{N}) + \delta(J_{i,j} + J/\sqrt{N}) \}. \tag{11}$$

Substituting Eq. (11) into Eqs. (8) and (9), the integral over $J_{i,j}$ is next performed:

$$\rho(\lambda) = -\frac{2}{N\pi} \mathrm{Im} \frac{\partial}{\partial \lambda} \lim_{n \to 0} \frac{1}{n} \left\{ \left( \frac{e^{i\pi/4}}{\pi^{1/2}} \right)^{Nn} \int_{-\infty}^{\infty} \prod_{i;\alpha} dx_{i;\alpha} \exp \right.$$
$$\left. \times \left[ -i\lambda \sum_{i;\alpha} (x_i^\alpha)^2 \right] \prod_{i<j} \left[ \cos \left( \frac{2J}{\sqrt{N}} \sum_\alpha x_i^\alpha x_j^\alpha \right) - 1 \right] \right\}. \tag{12}$$

Of course,

$$\prod_{i<j} \cos \left( \frac{2J}{\sqrt{N}} \sum_\alpha x_i^\alpha x_j^\alpha \right)$$
$$\equiv \exp \left\{ \frac{1}{2} \sum_{i,j} \ln \left[ \cos \left( \frac{2J}{\sqrt{N}} \sum_\alpha x_i^\alpha x_j^\alpha \right) \right] \right\}. \tag{13}$$

It should be pointed out, thanks to a comment by a reviewer, that the diagonal elements of the right-hand side in this transformation should be more thoroughly discussed; see the Appendix. Thereafter the argument of the exponential can be expanded in powers of $1/N$ to read

$$\prod_{i<j} \cos \left( \frac{2J}{\sqrt{N}} \sum_\alpha x_i^\alpha x_j^\alpha \right)$$
$$\simeq \exp \left\{ \sum_{i,j} \left[ \frac{-J^2}{N} \left( \sum_\alpha x_i^\alpha x_j^\alpha \right)^2 \right] \right.$$
$$\left. \times \left[ 1 + \frac{2J^2}{3N} \left( \sum_\alpha x_i^\alpha x_j^\alpha \right)^2 \right] + O(N^{-3}) \right\}. \tag{14}$$

Observe the term in brackets in Eq. (14). It will lead to the relevant term in Eq. (19) distinguishing a difference between matrices with Gaussian distributed matrix elements and those with the binary distribution considered here, and subsequently to Eqs. (34) and (38) for the $1/N$ correction to $\rho(\lambda)$.

Keeping only the first term of the exponential and following the JD analysis, the AED is easily obtained in the limit $N \to \infty$, i.e., a semicircle. To obtain the finite-size-$N$ case, the next leading term must be conserved:

$$
\frac{-J^2}{N} \sum_{i,j} \left( \sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right)^2
$$

$$
= \frac{-J^2}{N} \sum_{\alpha} \left[ \sum_i (x_i^{\alpha})^2 \right]^2 + \frac{-J^2}{N} \sum_{i,j} \left( \sum_{\alpha \neq \beta} x_i^{\alpha} x_j^{\alpha} x_i^{\beta} x_j^{\beta} \right). \tag{15}
$$

As in Edwards and Warner [52] and Dhesi and Jones [45], it can be shown that the first term is an order of magnitude higher than the second when $N \to \infty$. Indeed, even though each $\alpha \neq \beta$ terms tend to contribute to the AED, the sum over roman indices $i$ and $j$ reduces their input.

Nevertheless, when calculating the next contribution to the AED, the $\alpha \neq \beta$ terms must be conserved. However, the second term can be decomposed into $\alpha$, $\beta$, $\gamma$, $\delta$ contributions. Again, the $\alpha = \beta = \gamma = \delta$ terms will contribute to an order of magnitude larger value than those with nonequal indices. The full formal expression is not written for conciseness; see below for its practical evaluation [Eq. (18)].

## IV. PERTURBATION METHOD

In this section, the starting idea is to use a Hubbard–Stratonovich transformation (or the so-called auxiliary field identity) to express the second and third term in the exponential

$$
\exp \left\{ \frac{-J^2}{N} \left[ \sum_i (x_i^{\alpha})^2 \right]^2 \right\} = \frac{N^{1/2}}{(2\pi)^{1/2}} \frac{1}{(2J^2)^{1/2}} \lambda \int_{-\infty}^{\infty} ds^{\alpha} \exp \left[ -\frac{\lambda^2}{4J^2} N(s^{\alpha})^2 - i\lambda s^{\alpha} \sum_i (x_i^{\alpha})^2 \right], \tag{16}
$$

and similarly for

$$
\exp \left\{ \frac{-2J^4}{3N^2} \left[ \sum_i (x_i^{\alpha})^4 \right]^2 \right\} = \sqrt{\frac{3}{8\pi}} \int_{-\infty}^{\infty} dp^{\alpha} \exp \left[ -\frac{3}{8}(p^{\alpha})^2 - i\frac{J^2}{N} p^{\alpha} \sum_i (x_i^{\alpha})^4 \right]. \tag{17}
$$

Gathering all the relevant terms, the AED reads

$$
\rho(\lambda) = -\frac{-2}{N\pi} \frac{\partial}{\partial \lambda} \lim_{n \to 0} \frac{1}{n} \left\{ \left( \frac{e^{i\pi/4}}{\pi^{1/2}} \right)^{Nn} \int \prod_{\alpha} ds_{\alpha} \frac{N^{1/2}}{(2\pi)^{1/2}} \frac{1}{(2J^2)^{1/2}} \lambda \exp \left[ -\frac{\lambda^2}{4J^2} N \sum_{\alpha} (s^{\alpha})^2 \right] \right.
$$

$$
\left. \times \int \prod_{\alpha} dp_{\alpha} \sqrt{\frac{3}{8\pi}} \exp \left[ -\frac{3}{8} \sum_{\alpha} (p^{\alpha})^2 \right] L(s;p) - 1 \right\}, \tag{18}
$$

where

$$
L(s;p) = \int \prod_{i;\alpha} dx_{i;\alpha} \exp \left\{ -i\lambda \sum_{i;\alpha} (x_i^{\alpha})^2 - i\lambda s^{\alpha} \sum_{i;\alpha} (x_i^{\alpha})^2 - i\frac{J^2}{N} p^{\alpha} \sum_{i;\alpha} (x_i^{\alpha})^4 - \frac{J^2}{N} \left( \sum_{i,j;\alpha \neq \beta} x_i^{\alpha} x_j^{\alpha} x_i^{\beta} x_j^{\beta} \right) \right\}. \tag{19}
$$

From a close examination of Eq. (19), it can be noticed that the third and fourth terms in the exponential do *not* contribute to the AED in the $\lim N \to \infty$. Thus a "perturbation expansion" can be constructed in order to represent $L(s;p)$ as

$$
L(s;p) = \int \prod_{i;\alpha} dx_i^{\alpha} \left\{ \exp \left[ -i\lambda(1 + s^{\alpha}) \sum_{i;\alpha} (x_i^{\alpha})^2 \right] \right\} (1 + A)(1 + B). \tag{20}
$$

with

$$
A = -i\frac{J^2}{N} p^{\alpha} \sum_{i;\alpha} (x_i^{\alpha})^4 + \frac{1}{2!} \left( -i\frac{J^2}{N} p^{\alpha} \right)^2 \left( \sum_{i;\alpha} (x_i^{\alpha})^4 \right)^2 + \cdots, \tag{21}
$$

$$
B = \frac{-J^2}{N} \left( \sum_{i,j;\alpha \neq \beta} x_i^{\alpha} x_j^{\alpha} x_i^{\beta} x_j^{\beta} \right) + \frac{1}{2!} \left( \frac{-J^2}{N} \right)^2 \left( \sum_{i,j;\alpha \neq \beta} x_i^{\alpha} x_j^{\alpha} x_i^{\beta} x_j^{\beta} \right)^2 + \cdots. \tag{22}
$$

Next, consider the part of $L$ involving the $AB$ term, denoting it by $L_{AB}$, i.e.,

$$
L_{AB} = \int \prod_{i,\alpha} dx_i^{\alpha} \exp \left[ -i\lambda(1 + s^{\alpha}) \sum_{i;\alpha} x_i^{\alpha 2} \right] AB \equiv \sum_{c,d} L_{ABcd}, \tag{23}
$$

where $L_{ABcd}$ is the contribution to $L_{AB}$ from the $c$-th and $d$-th terms of $A$ and $B$ (where $c$ and $d$ are positive integers). We employ a diagrammatic technique [53] to evaluate $L_{AB}$ by representing

$$(x_k^\gamma)^4 \;\rightarrow\; \underset{k;\gamma}{} \qquad (24)$$

and

$$\underset{i,j\,;\alpha=\beta}{} x_i^\alpha x_j^\alpha x_i^\beta x_j^\beta \;\rightarrow\; \qquad \cdot \quad (25)$$

Thus, taking the first term from both $A$ and $B$ we can represent $L_{AB11}$ symbolically as

$$-i\frac{J^4}{N^2}p^\gamma \left\{ \qquad \right\}, \qquad (26)$$

where the brackets {} denote the average against the Gaussian weight in Eq. (23).

Following the usual diagrammatic technique plus bearing in mind that $\alpha \neq \beta$ we evaluate the integral defining $L_{AB}$ by contracting the legs. In so doing, we produce either connected or disconnects diagrams

### A. Disconnected diagram

The disconnected diagram reads

$$\qquad \cdot \qquad (27)$$

This diagram gives a contribution whose $n$-dependence is of the $O(n(n^2 - n))$. Remembering that we have to take the limit $n \to 0$, in evaluating the AED, it becomes clear (also see DJ [45]) that to produce a non zero contribution to the AED, we need to retain terms that are linear in $n$. Therefore the <u>above</u> diagram contributes nothing to the AED. It can be seen that a necessary condition for a diagram to be linear in $Nn$ is that it be connected. This result is general and holds to all orders in perturbation theory [54].

### B. Connected diagram

Bearing in mind that $\alpha \neq \beta$, the connected diagram that contributes of $O(1/N)$ to $L_{AB;11}$ is

$$\qquad \cdot \qquad (28)$$

Since $\alpha \neq \beta$, then $i = j = k$. This in turn provides a contribution to AED of $O(1/N^2)$ Furthermore, it is realized that any general term of $L_{ABcd}$ will give rise to connected diagrams that are linear in $n$ will be to a maximum to $O(N^{-1})$.

Subsequently $L_{AB}$ contributes to the AED to a maximum of $O(N^{-2})$. As we are to evaluate the AED to $O(N^{-1})$, this $L_{AB}$ part of $L$ can thus be disregarded.

Whence Eq. (19) can be rewritten as

$$L = I(1 + M + K), \qquad (29)$$

where

$$I = \int \prod_{i;\alpha} dx_{i;\alpha} \exp\left[ -i\lambda(1 + s^\alpha) \sum_{i;\alpha} \left(x_i^\alpha\right)^2 \right], \qquad (30)$$

$$M = \int \prod_{i;\alpha} dx_{i;\alpha} \exp\left[ -i\frac{J^2}{N} p^\alpha \sum_{i;\alpha} \left(x_i^\alpha\right)^4 \right], \qquad (31)$$

$$K = \int \prod_{i;\alpha} dx_{i;\alpha} \exp\left[ -i\frac{J^2}{N} p^\alpha \sum_{i,j;\alpha\neq\beta} x_i^\alpha x_j^\alpha x_i^\beta x_j^\beta \right]. \qquad (32)$$

Thus, Eq. (19) can be further rewritten to have the form

$$\rho(\lambda) = \rho^{(I)}(\lambda) + \rho^{(M)}(\lambda) + \rho^{(K)}(\lambda), \qquad (33)$$

in terms of the and $I$, $M$, and $K$ functions defined here above.

Further progress can be made in evaluating $I$, $M$, and $K$ to order unity such that $\rho^{(M)}(\lambda)$, $\rho^{(K)}(\lambda)$, and $\rho^{(I)}(\lambda)$ be known to $O(1/N)$, as follows:

i. evaluating $M$ [Eq. (31)], one finds to $O(1)$

$$M = \exp\left\{ nN\left[ \ln\left( \frac{\pi}{i\lambda(1 + s^\alpha)} \right) \right]^{1/2} \right. $$
$$\left. + \ln\left[ 1 - i\frac{J^2}{N} p^\alpha \frac{3}{[2i\lambda(1 + s^\alpha)]^2} \right] \right\}. \qquad (34)$$

When substituting the $O(1)$ term into Eq. (31), by assuming replica symmetry and using the identity Eq. (5) to reconstruct the logarithm, we obtain

$$\rho^{(M)}(\lambda) = \frac{-2}{N\pi} \text{Im} \frac{\partial}{\partial\lambda} \ln\left\{ \left( \frac{e^{i\pi/4}}{\pi^{1/2}} \right)^N \left( \frac{N}{2\pi} \right)^{1/2} \frac{\lambda}{(2J^2)^{1/2}} \right.$$
$$\times \int ds \exp\left[ -\frac{\lambda^2}{4J^2} Ns^2 + N\ln\left( \frac{\pi}{i\lambda(1 + s)} \right)^{1/2} \right]$$
$$\times \int dp \left( \frac{3}{8\pi} \right)^{1/2} \exp$$
$$\left. \times \left[ \frac{-3}{8} p^2 - iJ^2 p \frac{3}{(2i\lambda(1 + s))^2} \right] \right\}. \qquad (35)$$

The $p$ integral in Eq. (35) contains only an exponential and can be easily evaluated. Thus, $\rho^{(M)}(\lambda)$ takes the concise form

$$\rho^{(M)}(\lambda) = \frac{-2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\ln\left\{\lambda\exp\left(\frac{-N}{2}\ln\lambda\right)\right.$$
$$\left.\times\int ds\exp[-Ng(s)+h(s)]\right\}, \quad (36)$$

where

$$g(s) = \frac{\lambda^2 s^2}{4J^2} + \frac{1}{2}\ln[i(1+s)], \quad (37)$$

$$h(s) = -\frac{3}{8}J^4\frac{1}{[i\lambda(1+s)]^4}. \quad (38)$$

ii. A similar procedure goes for evaluating $K$ [Eq. (32)] (see also DJ [45]). Notice that there is no $p$ variable in $K$, whence the $p$ integral $= 1$. Thus,

$$\rho^{(K)}(\lambda) = \frac{-2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\ln\left\{\lambda\exp\left(\frac{-N}{2}\ln\lambda\right)\right.$$
$$\left.\times\int ds\exp[-Ng(s)+f(s)]\right\}, \quad (39)$$

where

$$f(s) = \frac{1}{4}\ln\left[1 - \frac{J^2}{\lambda^2(1+s)^2}\right]. \quad (40)$$

iii. Evaluating $I$ [Eq. (30)] is more straightforward; thereafter, one obtains

$$\rho^{(I)}(\lambda) = \frac{-2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\ln\left\{\lambda\exp\left(\frac{-N}{2}\ln\lambda\right)\right.$$
$$\left.\times\int ds\exp[-Ng(s)]\right\}. \quad (41)$$

The above results can be easily collected to rewrite $\rho(\lambda)$ [Eq. (33)] exactly to $O(1/N)$.

## V. AVERAGED EIGENVALUE DENSITY AND GREEN'S FUNCTION TO $O(1/N)$

The method of steepest descent allows us to obtain the AED to $O(1/N)$. The interesting saddle point $s_0$ associated with $g(s)$

is found from

$$\left.\frac{\partial g}{\partial s}\right|_{s=s_0} = 0. \quad (42)$$

Thus,

$$\rho^{(M)}(\lambda) = \frac{-2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\left\{-\frac{N}{2}\ln\lambda - Ng(s_0) + \ln\lambda\right.$$
$$\left.-\frac{1}{2}\ln g''(s_0) + h(s_0)\right\}. \quad (43)$$

Similar expressions are obtained for $\rho^{(N)}$ and $\rho^{(I)}$, when replacing $h(s_0)$ by $f(s_0)$ and 1, respectively, in Eq. (43).

In other words, to $O(1/N)$, one has

$$\rho(\lambda) = \rho_0(\lambda) + \rho_{1/N}(\lambda) \quad (44)$$

with

$$\rho_0(\lambda) = \frac{-2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\ln\left\{-\frac{N}{2}\ln\lambda - Ng(s_0)\right\}, \quad (45)$$

$$\rho_{1/N}(\lambda) = \frac{-2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\ln\left\{\lambda - \frac{1}{2}\ln g''(s_0) + h(s_0) + f(s_0)\right\}. \quad (46)$$

From the definition Eq. (38), it is easy to see that $g(s;\lambda)$ has two complex conjugate saddle points at

$$\tfrac{1}{2}\{-1 \mp i[1 - (4J^2/\lambda^2)]\}^{1/2}.$$

It has been argued in Edwards and Jones [31] that the contour chosen for the saddle point approximation may only be deformed to pass through one of these saddle points, and that the lower saddle point leads to a physically reasonable positive AED. Thus, following Edward and Jones [31], we choose the $-i$ sign saddle point in the above expression to go on.

We now explicitly evaluate $\rho(\lambda)$. The contribution $\rho_0(\lambda)$, obtained from Eq. (45) and taking into account Eq. (37), is

$$\rho_0(\lambda) = \frac{1}{\pi}\mathrm{Im}\left\{\frac{1}{2J^2}[\lambda + i(4J^2 - \lambda^2)^{1/2}]\right\}, \quad (47)$$

yielding the corresponding Green's function

$$G_0(\lambda) = \frac{1}{2J^2}[\lambda + i(4J^2 - \lambda^2)^{1/2}], \quad (48)$$

thereby proving that

$$\rho_0(\lambda) = \begin{cases} \left(\frac{1}{2\pi J^2}\right)[4J^2 - \lambda^2]^{1/2} & \text{for } |\lambda| < 2J, \\ 0 & \text{for } |\lambda| > 2J. \end{cases} \quad (49)$$

The first-order correction $\rho_{1/N}(\lambda)$ is obtained after some simple algebra, taking into account Eqs. (38) and (40), and reads

$$\rho_{1/N}(\lambda) = \begin{cases} \frac{1}{4N}[\delta(\lambda+2J) + \delta(\lambda-2J)] - \frac{1}{2N\pi}\frac{1}{[4J^2-\lambda^2]^{1/2}} + \frac{3}{N\pi}\frac{[4J^2-\lambda^2]^{1/2}}{8J^4}\{[3\lambda^2 - 2J^2] - \frac{2\lambda^2(\lambda^2-2J^2)}{[4J^2-\lambda^2]}\} & \text{for } |\lambda| < 2J \\ 0 & \text{for } |\lambda| > 2J. \end{cases} \quad (50)$$

Thereafter, the first-order Green's function correction $G_1(\lambda)$ is immediately obtained from $\rho_{1/N}(\lambda)$. Moreover, the

former and the latter functions can be expressed in terms of the zeroth-order Green's function or AED, respectively. After

some lengthy but simple algebra, it is found that

$$G(\lambda) = G_0\left\{1 + \frac{1}{N}\left[\frac{J^2 G_0^2}{\left(1 - J^2 G_0^2\right)^2} - 3\frac{(JG_0)^4}{1 - J^2 G_0^2}\right]\right\}, \quad (51)$$

where the variable $(\lambda)$ has not be written on the right-hand side, and $G_0$ is defined in Eq. (48).

Equations (49)–(51) are the new intended results, whence presenting the extra terms not found in previously treated cases, e.g., Ref. [45], arising from the symmetry and sign of the element constraint imposed on the Wigner matrix.

## VI. COMMENTS

For a short discussion, let us decompose $\rho_{1/N}(\lambda)$ such that

$$\rho_{1/N}(\lambda) = \rho_{1/N}^{(Q)}(\lambda) + \rho_{1/N}^{(R)}(\lambda), \quad (52)$$

thus, where

$$\rho_{1/N}^{(Q)}(\lambda) = \frac{1}{4N}[\delta(\lambda + 2J) + \delta(\lambda - 2J)] - \left\{\frac{1}{2N\pi}\frac{1}{[4J^2 - \lambda^2]^{1/2}}\right\}, \quad (53)$$

which is identical to Eq. (4.14) of DJ [45] and

$$\rho_{1/N}^{(R)}(\lambda) = \frac{3}{N\pi}\frac{[4J^2 - \lambda^2]^{1/2}}{8J^4}\left\{[3\lambda^2 - 2J^2] - \frac{2\lambda^2(\lambda^2 - 2J^2)}{[4J^2 - \lambda^2]}\right\}. \quad (54)$$

Equation (54) is the extra term coming from the Feynman diagrams presented here above, which added to the Eq. (52) corresponding to the $O(1/N)$ correction to the Gaussian Orthogonal Ensemble distribution, is one key point of our present work. Here, it becomes clear that the first correction for the RSSME arises from $\rho_{1/N}(\lambda)$, which in turn is based on the function $h(s)$ defined by Eq. (38).

On Figs. 2–5, we display $\rho_0(\lambda)$ and $\rho(\lambda)$ for $N = 2$, 10, 20, and 200 respectively. For convenience $J$ is taken to be $=1$. We notice that there is a significant departure of $\rho(\lambda)$ from the semicircle $\rho_0(\lambda)$ even for values of $N$ as large as 20. For the GOE, the departure from the semicircle becomes noticeable only for N <6 [41,43]. By comparison with the figures supplied in DJ [45], it can be seen that the significant departure is due to $\rho_{1/N}^{(R)}(\lambda)$ rather than $\rho_{1/N}^{(Q)}(\lambda)$. From this we deduce that it is this new correction $\rho_{1/N}^{(R)}(\lambda)$ together with $\rho_0(\lambda)$ which mimics the broadening of the two mirror imaged Poisson type distributions as $N$ becomes large. In the limit of
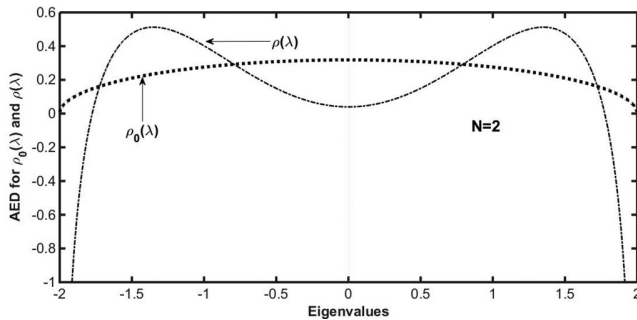


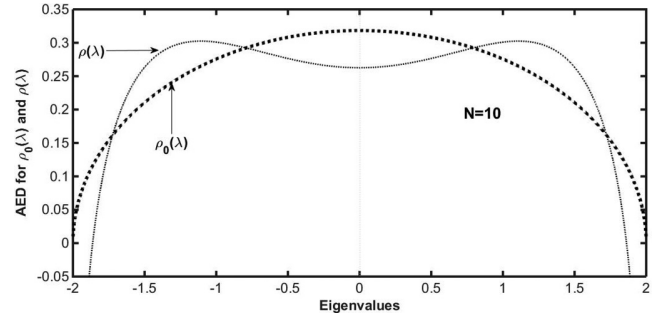FIG. 3. Plot of $\rho_0(\lambda)$ and $\rho(\lambda)$ for $N = 10$.

large $N$ ($\sim$200), this broadening and overlapping tend to the semicircle. One should not expect $\rho(\lambda)$ to mimic correctly the AED of the RSSME when $N < 6$. Corrections to $O(1/N^2)$ will be required for these low values of $1/N$. They will also be required when mimicking the fine structure of the spectrum.

The displayed figures also bring to the fore that $\rho_{1/N}(\lambda)$ possesses divergences near the band edges $|\lambda| = 2J$ of the semicircle. This is not surprising for reasons mentioned earlier. Thus, result $\rho_0(\lambda)$, combining Eq. (49) and Eq. (50), should only be considered as best away and inside the band edges.

Briefly, we finally comment on the result in Eq. (51). Similarly we decompose $G_1(\lambda)$, as done for $\rho_{1/N}(\lambda)$, into

$$G_1(\lambda) = G_1^{(Q)}(\lambda) + G_1^{(R)}(\lambda), \quad (55)$$

whence with

$$G_1^{(Q)}(\lambda) = \frac{1}{N}\left[\frac{(JG_0)G_0(JG_0)}{[1 - (JG_0)^2]^2}\right], \quad (56)$$

and

$$G_1^{(R)}(\lambda) = -\frac{3}{N}\left[\frac{(JG_0)^2 G_0(JG_0)^2}{[1 - (JG_0)^2]}\right]. \quad (57)$$

It can be noticed that $G_1^{(Q)}(\lambda)$ is the first-order correction of the GOE, while $G_1^{(R)}(\lambda)$ is the newly found first-order correction to the RSSME.

## VII. NUMERICAL SIMULATIONS

Numerical simulations proceeded as follows: first, the matrix size $N$ is decided upon, and zeros are put on the diagonal. Next, one picks at random an element $a_{i,j}$ of the matrix, with $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant N$; one attributes either



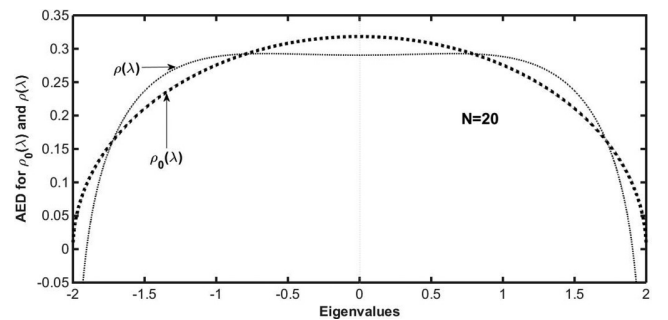FIG. 2. Plot of $\rho_0(\lambda)$ and $\rho(\lambda)$ for $N = 2$.



FIG. 4. Plot of $\rho_0(\lambda)$ and $\rho(\lambda)$ for $N = 20$.
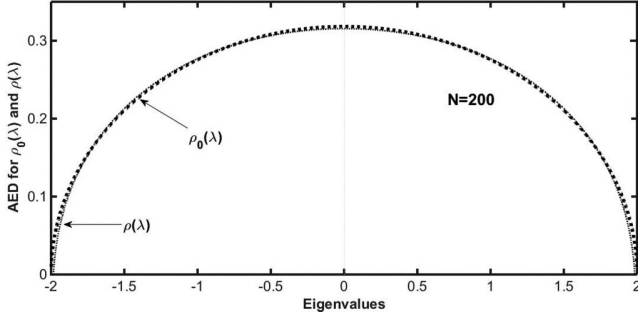
FIG. 5. Plot of $\rho_0(\lambda)$ and $\rho(\lambda)$ for $N = 200$. The curves are hardly distinguishable.

the value $+J/\sqrt{N}$ or $-J/\sqrt{N}$ with equal probability to $a_{i,j}$ *and* to $a_{j,i}$ (one can take $J = 1$ without loss of generality). Do such an attribution for another $a_{i,j}$ element; in fact successively for all $N(N-1)$ elements of the matrix. Calculate the $N$ eigenvalues and store them. Repeat the matrix construction a large number of times. In the present case, all $N$-size matrices were invented a million of times, except for $N = 200$, which was invented only 50 000 times.

The histogram of eigenvalues for the set of $N$-given finite size matrices is thus obtained. Practically, the histogram is normalized according to the number of simulations, to obtain the "averaged spectrum."

The display of such an AED is shown in Fig. 1 and Figs. 6–8. On such figures, the theoretically obtained first-order finite size correction $\rho(\lambda)$ is also given for comparison.

Noticed that each numerical spectrum seems to have some nice band tails.

## VIII. CONCLUSIONS

By using the replica method, we searched for the first finite size $O(1/N)$ correction to the averaged eigenvalue density *and* to the corresponding Green's function of a random sign symmetric matrix ensemble. It is well known that the AED of a $d$-regular random graph with $N$ vertices, in the limit $N \to \infty$ and $d \to \infty$, obeys the Kesten–McKay law [35,36]. However, fully random systems are only theoretical cases of interest. Thus it seemed worthwhile to calculate correction terms to the AED in view of handling more realistic systems. In our work,
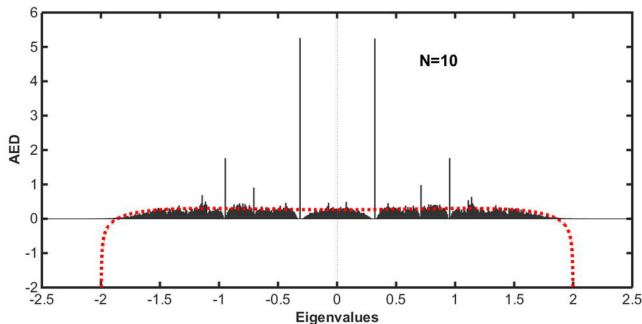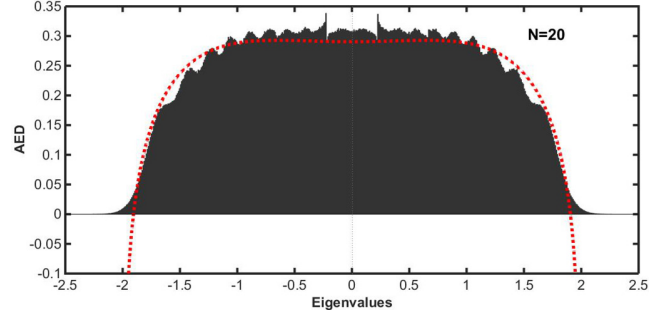


FIG. 7. Comparison of the numerical simulation AED (vertical lines, delta functions) with the theoretical first-order $O(1/N)$ approximation (red) dotted line for $N = 20$.

the former correction term to $O(1/N)$ becomes written as in Eq. (50), while the total Green's function correction term reads as in Eq. (51):

$$G_1(\lambda) = G_0 \frac{1}{N} \left[ \frac{J^2 G_0^2}{\left(1 - J^2 G_0^2\right)^2} - 3 \frac{(J G_0)^4}{1 - J^2 G_0^2} \right]. \quad (58)$$

The interpretation of the extra terms seems rather clear: it pertains to the reduced number of "degrees of freedom" of the system, within the Hamiltonian and the corresponding matrix of (a reduced number of) possible states; the "restriction" being found in the equal probability condition for the *binary* distribution of matrix elements, but in the "extension" in the $\pm$ sign of these matrix elements.

It can be usefully reinstated that the term in brackets in Eq. (14) leads to the relevant term in Eq. (19), thereby allowing us to distinguish the difference between matrices with Gaussian distributed matrix elements and those with a binary distribution considered in this paper. This term subsequently sustains Eqs. (34) and (38) for the $1/N$ correction to $\rho(\lambda)$, whence going beyond the DJ analysis [45].

Beside analytic works, simulation results are presented. The comparison between the analytical formulas and the numerical diagonalization results for finite size matrices exhibits an excellent agreement, confirming the correctness of the first-order finite size expression.

It has been emphasized that the $1/N$ corrections of the AED diverge at the band edges of the semicircle. This



FIG. 6. Comparison of the numerical simulation AED (vertical lines, delta functions) with the theoretical first-order $O(1/N)$ approximation (red) dotted line for $N = 10$.
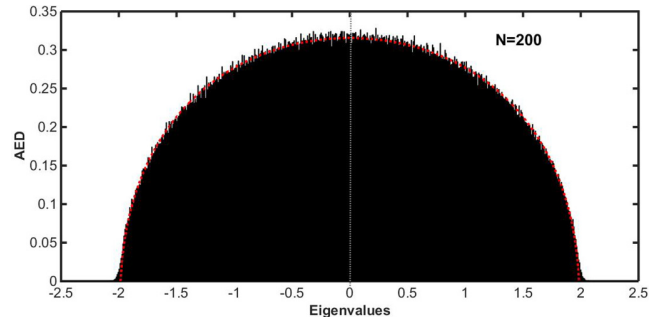


FIG. 8. Comparison of the numerical simulation AED (blue) vertical lines, delta functions, with the theoretical first-order $O(1/N)$ approximation (red) dotted line for $N = 200$.

"problem" should be considered in further work. Some self-consistency condition imposed on the diagrammatic formalism should produce a finite AED throughout the whole spectrum. However, this is obviously outside the present aim.

More recently, studies of the properties of random matrices have found a new revitalization due to the mapping of networks and graphs through their adjacency matrix. In these cases, an additional input stems from the possible directionality of the link or bond. Let us have in mind, for argument, the case of a citation or any type of cooperation and/or competition network. Due to the intrinsically time-dependent hierarchical process, the adjacency matrix representing the network is usually asymmetric, beside being a non-negative matrix. This asymmetry, leading to complex eigenvalues, significantly widens the realm of investigations [55,56]. The asymmetry case is *not* treated in the present paper but is mentioned in this conclusion section, for any reader guideline interested in pursuing the present work.

Thus, further work, if we may suggest so, should be programed to apply the approach in view of obtaining results on disordered systems characterized, e.g., (i) by symmetric matrices having more complicated structures, as in financial or socio-economic matters [55–57], or (ii) when more than one type of disorder appears, as is the case very often in materials [58–60], and (iii) by nonsymmetric matrices: see the cases of citation or co-authorship networks implying link ordering [61–63], that of bipartite graphs [64], that of physiology [65], or that of financial markets [66–70], among recent relevant cases.

### APPENDIX: ON "NEGLECTING" DIAGONAL TERMS IN THE RIGHT-HAND SIDE OF EQ. (13)

Recall Eq. (13) transforming a product of cosines into a product of exponentials, apparently including diagonal ($i = j$) terms on the right-hand side, but not including them on the right-hand side. Although this should be much incorrect indeed on rigorous grounds, let it be recalled that

$$\ln\left[\cos\frac{A}{\sqrt{(N)}}\right] \simeq \ln\left[1 - \frac{A^2}{2N} + \cdots\right] \simeq -\frac{A^2}{2N} + O\left(\frac{1}{N^2}\right), \tag{A1}$$

to leading order in $N$. Therefore,

$$\rho(\lambda) = -\frac{2}{N\pi}\mathrm{Im}\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left\{\left(\frac{e^{i\pi/4}}{\pi^{1/2}}\right)^{Nn}\int_{-\infty}^{\infty}\prod_{i;\alpha}dx_{i;\alpha}\right.$$
$$\times\left[\exp\left(-i\lambda\sum_{i;\alpha}(x_i^\alpha)^2\right)\right]$$
$$\times\left.\prod_{i<j}\left[\cos\left(\frac{2J}{\sqrt{N}}\sum_\alpha x_i^\alpha x_j^\alpha\right) - 1\right]\right\} \tag{A2}$$

can be rewritten as in Eq. (14)

$$\prod_{i<j}\cos\left(\frac{2J}{\sqrt{N}}\sum_\alpha x_i^\alpha x_j^\alpha\right)$$
$$\simeq \exp\left\{\sum_{i,j}\left[\frac{-J^2}{N}\left(\sum_\alpha x_i^\alpha x_j^\alpha\right)^2\right]\right.$$
$$\times\left.\left[1 + \frac{2J^2}{3N}\left(\sum_\alpha x_i^\alpha x_j^\alpha\right)^2\right] + O(N^{-3})\right\}, \tag{A3}$$

when neglecting the contribution of the diagonal terms. These read

$$\sum_{i,j}\left[\frac{-J^2}{N}\left(\sum_\alpha (x_i^\alpha)^2\right)^2\right]\left[1 + \frac{2J^2}{3N}\left(\sum_\alpha (x_i^\alpha)^2\right)^2\right]. \tag{A4}$$

The first term

$$\simeq \sum_i\left[\left(\sum_\alpha (x_i^\alpha)^2\right)^2\right] \tag{A5}$$

and the second term (coming for the consideration of finite size effects)

$$\simeq \sum_i\left[\left(\sum_\alpha (x_i^\alpha)^2\right)^4\right] \tag{A6}$$

give a contribution $O(n^2)$ and $O(n^4)$, respectively, in the limit $n \to 0$. Thus the diagonal elements in Eq. (14), after the transformation resulting from the expansion (13), can be neglected in the limit $n \to 0$.

Notice that, if we had started with the ensemble described by Wigner, i.e., the diagonal elements equal to zero, we should have arrived at Eq. (61) for $\rho(\lambda)$ as well. In this sense, it is even irrelevant whether the diagonal elements are equal to zero.

[1] T. Tao, *Topics in Random Matrix Theory*, Graduate Studies in Mathematics (American Mathematical Society, Providence, RI, 2012), Vol. 132.

[2] E. Wigner, Ann. Math. **67**, 325 (1958).

[3] E. Wigner, SIAM Rev. **9**, 1 (1967).

[4] E. N. Foo and M. Ausloos, J. Non-Cryst. Solids **8–10**, 134 (1972).

[5] M. L. Mehta and M. Gaudin, Nucl. Phys. **18**, 420 (1960).

[6] B. Kozma and G. Korniss, in *Computer Simulation Studies in Condensed Matter Physics XVI*, edited by D. P. Landau, S. P. Lewis, and H.-B. Schuttler (Springer-Verlag, Berlin, 2004), pp. 29–33.

[7] F. Slanina, Eur. Phys. J. B **85**, 1 (2012).

[8] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11487 (1993).

[9] G. Parisi, F. Ritort, and F. Slanina, J. Phys. A: Math. Gen. **26**, 3775 (1993).

[10] O. Perron, Math. Ann. **64**, 248 (1907).

[11] G. Frobenius, in *Uber Matrizen aus nicht Negativen Elementen* (S.-B. Prüss. Acad. Wiss., Berlin, 1912), pp. 456–477.

[12] R. B. Bapat and T. E. S. Raghavan, *Nonnegative Matrices and Applications, Encyclopedia of Mathematics and its Applications* (Cambridge University Press, Cambridge, 1997), Vol 64.

[13] A. Berman, M. Neumann, and R. J. Stern, *Nonnegative Matrices in Dynamic Systems* (John Wiley & Sons, New York, 1989).

[14] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics* (SIAM, Philadelphia, 1994).

[15] H. Minc, *Nonnegative Matrices* (Wiley, New York, 1988).

[16] U. G. Rothblum, in *Handbook of Linear Algebra*, edited by L. Hogben (Chapman & Hall/CRC Press, Boca Raton, 2007).

[17] E. Senata, *Nonnegative Matrices and Markov Chains*, Springer Series in Statistics (Springer-Verlag, New York, 1981).

[18] A graph is called regular if every vertex has the same degree, and $d$-regular if that degree is $d$; a $d$-random regular graph has a mean degree equal to $d$. See D. Spielman, *Combinatorial Scientific Computing* (Chapman and Hall/CRC Press, Boca Raton, 2012), www.cs.yale.edu/spielman/PAPERS/SGTChapter.pdf and Ref. [19].

[19] C. Lucibello, F. Morone, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo, Phys. Rev. E **90**, 012146 (2014).

[20] C. W. J. Beenakker, Rev. Mod. Phys. **87**, 1037 (2015).

[21] G. Le Caer and R. Delannay, J. Phys. A: Math. Theor. **40**, 1561 (2007).

[22] R. Delannay and G. Le Caer, Phys. Rev. E **62**, 1526 (2000).

[23] J. M. Nieminen, J. Phys. A: Math. Theor. **42**, 035001 (2009).

[24] E. Bogomolny and J. P. Keating, J. Phys. A: Math. Theor. **46**, 305203 (2013).

[25] E. P. Wigner, Ann. Math. Lpz. **67**, 226 (1958).

[26] C. E. Porter, *Statistical Theories of Spectra: Fluctuations* (Academic Press, New York, 1965).

[27] M. Mehta, *Random Matrices and the Statistical Theory of Energy Levels* (Academic Press, New York, 1967).

[28] L. Pastur and M. Shcherbina, *Eigenvalue Distribution of Large Random Matrices*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, RI, 2011), Vol. 171.

[29] P. Deift, *Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert Approach*, Courant Lecture Notes (American Mathematical Society, Providence, 2000), Vol. 3.

[30] W. König, Probab. Surv. **2**, 385 (2005).

[31] S. F. Edwards and R. C. Jones, J. Phys. A: Math. Gen. **9**, 1595 (1976).

[32] S. F. Edwards, in *Proceedings of the 4th International Conference on Amorphous Materials*, edited by R. W. Douglas and B. Ellis (Wiley, New York, 1970), Chap. 30.

[33] M. Advani, S. Lahiri, and S. Ganguli, J. Stat. Mech. (2013) P03014.

[34] E. P. Wigner, Ann. Math. **62**, 548 (1955).

[35] H. Kesten, Trans. Am. Math. Soc. **92**, 336 (1959).

[36] B. McKay, Lin. Alg. Appl. **40**, 203 (1981).

[37] D. Jakobson, D. S. Miller, I. Rivin, and Z. Rudnick, in *Emerging Applications of Number Theory* (Springer, Berlin, 1999), pp. 317–327.

[38] I. Neri and F. L. Metz, Phys. Rev. Lett. **109**, 030602 (2012).

[39] L. V. Tran, V. H. Vu, and K. Wang, Random Struct. Algor. **42**, 110 (2013).

[40] R. Bauerschmidt, A. Knowles, and H. T. Yau, arXiv:1503.08702.

[41] R. C. Jones and G. S. Dhesi, Can. J. Phys. **68**, 1304 (1990).

[42] F. Takano and H. Takano, J. Phys. Soc. Jpn. **53**, 2943 (1984).

[43] J. J. M. Verbaarschot and M. R. Zirnbauer, Ann. Phys. (NY) **158**, 78 (1984).

[44] J. J. M. Verbaarschot, H. A. Weidenmuller, and M. R. Zirnbauer, Ann. Phys. (NY) **153**, 367 (1984).

[45] G. S. Dhesi and R. C. Jones, J. Phys. A: Math. Gen. **23**, 5577 (1990).

[46] F. L. Metz, G. Parisi, and L. Leuzzi, Phys. Rev. E **90**, 052109 (2014).

[47] E. Kanzieper and G. Akemann, Phys. Rev. Lett. **95**, 230201 (2005).

[48] J. Kwapień and S. Drożdż, Phys. Rep. **515**, 115 (2012).

[49] C. E. Porter and N. Rosenzweig, Ann. Acad. Sci. Fennicae. Ser. A **VI**, 3 (1960).

[50] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, New York, 1964).

[51] R. R. Nadakuditi and M. E. J. Newman, Phys. Rev. E **87**, 012803 (2013).

[52] S. F. Edwards and M. Warner, J. Phys. A: Math. Gen. **13**, 381 (1980).

[53] In a similar context, Feynman-like diagrams have been used to calculate the statistical quantities of embedded many-body random matrix problems by R. A. Small and S. Müller, Phys. Rev. E **90**, 010102(R) (2014).

[54] G. Grinstein and A. Luther, Phys. Rev. B **13**, 1329 (1976).

[55] G. Rotundo and M. Ausloos, Phys. A (Amsterdam, Neth.) **389**, 5479 (2010).

[56] G. Rotundo and M. Ausloos, Eur. Phys. J. B **86**, 1 (2013).

[57] M. Ausloos and M. Gligor, Acta Phys. Pol. A **114**, 491 (2008).

[58] E. N. Foo, H. Amar, and M. Ausloos, Phys. Rev. B **4**, 3350 (1971).

[59] *Superconductivity in Ternary Compounds II, Superconductivity and Magnetism*, Topics in Current Physics, edited by M. B. Maple and O. Fischer (Springer, Berlin, 1982), Vol. 34.

[60] H. G. Dill, Earth-Sci. Rev. **100**, 1 (2010).

[61] J. Kim and J. Diesner, Scientometrics **101**, 587 (2014).

[62] L. Wardil and Ch. Hauert, Phys. Rev. E **91**, 012825 (2015).

[63] B. Karrer and M. E. J. Newman, Phys. Rev. E **80**, 046110 (2009).

[64] R. Lambiotte and M. Ausloos, J. Stat. Mech. (2007) P08026.

[65] J. Kwapień, S. Drożdż, and A. A. Ioannides, Phys. Rev. E **62**, 5557 (2000).

[66] V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. Nunes Amaral, and H. E. Stanley, Phys. Rev. Lett. **83**, 1471 (1999).

[67] A. Utsugi, K. Ino, and M. Oshikawa, Phys. Rev. E **70**, 026110 (2004).

[68] J. Kwapień, S. Drożdż, A. Z. Gorski, and P. Oswiecimka, Acta Phys. Pol. B **37**, 3039 (2006).

[69] G. Livan and L. Rebecchi, Eur. Phys. J. B **85**, 213 (2012).

[70] See in particular Fig. 1(g) in C. Biely and S. Thurner, Quantum Finance **8**, 705 (2008).