Line-Point Constraints and Robot Surgery

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Abstract The space of rigid-body displacements that move a line so that its remains in contact with a fixed point is studied. This constraint variety is related to robot surgery where a straight, rigid cannula is inserted into the patient through a trocar. A surgical robot manipulates the cannula so the insertion point is fixed. The space of displacements determined by a pair of these constraints is also studied briefly. This correspond to a pair cannulas with their ends rigidly connected.

1 Introduction

This work is inspired by robot surgery, however the subject of this article is the geometry of robots. In robot surgery, tools are introduced into the patient's body through a trocar or "port". This port is a point on the patient's body through which the robot inserts a long thin cannula, the robot can also rotate the cannula about the point defined by the port. The rigid-body motion of a cannula is thus subject to a geometric constraint. The line defined by the cannula must pass through the point specified by the port. See [2] for a brief review of the history of this subject from the point of view of kinematics.

We begin with some mathematical background. The use of dual quaternions to represent rigid-body displacements dates back to E. Study at the beginning of the 20th century and is well known, see for example [3]. A rigid-body displacement can be represented in two ways by dual quaternions. A dual quaternion has the form,

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k).$$

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Here, $i,\ j$ and k form a base for the quaternions satisfying the usual rules for quaternions $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$ and so forth. The element ε is the dual unit which commutes with all quaterions but squares to zero, $\varepsilon^2 = 0$. The Study parameters, $a_0,\ a_1,\dots,c_3$, are real numbers. Not all dual quaterions represent rigid-body displacements. Taking only the dual quaternions satisfying the equations,

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \tag{1}$$

and

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0 \tag{2}$$

gives elements that comprise the double cover of the group of rigid-body displacements. This is the group $\mathrm{Spin}(3) \rtimes \mathbb{R}^3$. Both, g and $-g$ give the same rigid-body displacement. The elements satisfying these two equations form an affine algebraic variety.

Alternatively, we can think of the Study parameters as homogeneous coordinates in a 7-dimensional projective space $\mathbb{P}^7$. This has the effect of identifying the elements g and $-g$ since they only differ by multiplication by -1, a non-zero constant. Equation (1) cannot apply to these elements as it is not homogeneous. Equation (2) does apply, and dual quaternions satisfying this homogeneous equations form a 6-dimensional projective quadric variety usually known as the Study quadric. (A quadric is simply a variety of degree 2). Every rigid-body displacement corresponds to a unique dual quaternion in the Study quadric. Some elements of the Study quadric however, are not rigid-body displacements. These elements lie on the 3-dimensional plane lying entirely in the Study quadric and are defined by the equations $a_0 = a_1 = a_2 = a_3 = 0$. This 3-plane will be referred to as A_∞.

Away from A_∞, dual quaternions in the Study quadric can be written as,

$$g = (a_0 + a_1i + a_2j + a_3k) + \varepsilon(c_0 + c_1i + c_2j + c_3k) = r + \varepsilon\frac{1}{2}tr$$

where r is an ordinary quaternion representing a pure rotation and $t = t_xi + t_yj + t_zk$ is a pure quaternion representing a translation with translation vector $\vec{t} = (t_x,\ t_y,\ t_z)^T$. The action of such a rigid-body displacement on a point $(x,\ y,\ z)$ is given by the product,

$$\Big(r + \varepsilon\frac{1}{2}tr\Big)\Big(1 + \varepsilon(xi + yj + zk)\Big)\Big(r^- + \varepsilon\frac{1}{2}r^-t\Big) = (rr^-)\Big(1 + \varepsilon\big(rxi + yj + zk)r^- + t\big)\Big)$$

where $^-$ denotes the quaternion conjugate so that rr^- is a real scalar. Further, details can be found in several standard texts including [3].

In algebraic geometry the concepts of Veronese and Segre embeddings are fundamental. The Veronese embedding maps a projective space $\mathbb{P}^n$ into a projective space of larger dimension. Consider the n-dimensional projective space $\mathbb{P}^n$ with homogeneous $n + 1$ coordinates, $x_0,\ x_1,\dots,x_n$. The image of the degree d-Veronese embedding is parameterised by all $\binom{n+d-1}{d}$ degree d monomials in the x_i coordinates. That is, the image lies in $\mathbb{P}^m$ with coordinates $y_0,\ y_1,\dots,y_m$ where $m = \binom{n+d-1}{d} - 1$

and $y_0 = x_0^d,\ y_1 = x_0^{n-1}x_1, \dots, y_m = x_n^d$. The image of the this map is known as a Veronese variety, it can be shown to lie on a number of quadric hypersurfaces in $\mathbb{P}^m$. The simplest example is the degree 2 embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. This Veronese variety is a conic curve parameterised as $(x_0^2 : x_0x_1 : x_1^2)$ which is clearly the conic determined by the homogeneous equation $y_0y_2 - y_1^2 = 0$. Veronese varieties are generally not complete intersections, so Bézout's theorem cannot give their degree. However, it is possible to find the degree of a general Veronese variety, in [1, p.231] it is shown that the degree of the degree d embedding of $\mathbb{P}^n$ is d^n.

The Segre embedding maps the Cartesian product of projective spaces into another projective space. If the original projective spaces are $\mathbb{P}^n \times \mathbb{P}^m$ with coordinates x_i and y_j then the image of the Segre embedding, is parameterised by the possible products of pairs of coordinates. If z_k are the homogeneous coordinates of a $\mathbb{P}^q$ where $q = (n+1)(m+1) - 1 = nm + n + m$, then the image of the Segre embedding, the Segre variety, is parameterised by $z_0 = x_0y_0,\ z_1 = x_0y_1, \dots, z_q = x_ny_m$. The simplest example here is the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, given by, $z_0 = x_0y_0,\ z_1 = x_0y_1,\ z_2 = x_1y_0$ and $z_3 = x_1y_1$. This Segre variety is given by the quadric surface $z_0z_3 - z_1z_2 = 0$. In general, Segre varieties are not complete intersections but lie on several quadric hypersurfaces. The degree of the Segre variety $\mathbb{P}^n \times \mathbb{P}^m$ is $\binom{n+m}{n}$, see [1, p.233].

2 The Displacement Variety

In [6] the problem of finding the subvariety of displacements that move a point so that it remains in contact with a fixed plane was studied. Here the inverse problem of how to move a plane so that is remains in contact with a fixed point is addressed first. In [7], it was shown that reversing the order of an open loop kinemetic chain inverts the displacements that the end-effector can perform. If the variety determined by the possible displacements is given implicitly by equations in the Study parameters, then the equations satisfied by the inverses can be found by changing the signs of the coefficients which multiply odd numbers of the parameters $a_1,\ a_2,\ a_3$ and $c_1,\ c_2,\ c_3$.

Here we begin by looking at the varieties defined in terms other representations of the group of rigid-body displacements. Suppose we represent the fixed point by the extended vector,

$$\tilde{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} \quad \text{and the plane by the 4-vector,} \quad \pi = \begin{pmatrix} n_x \\ x_y \\ n_z \\ -d \end{pmatrix},$$

where $n_x,\ n_y$ and n_z are the components of the unit normal vector to the plane and d is the perpendicular distance from the plane to the origin. If the plane contains the point we have the following relation,

$$\tilde{p}^T \pi = p_xn_x + p_yn_y + p_zn_z - d = 0.$$

The action of the group of rigid-body displacements on the plane is given by the inverse-transpose of the standard 4×4 representation of $SE(3)$,

$$\pi' = \begin{pmatrix} R & 0 \\ -\vec{t}^{\,T} R & 1 \end{pmatrix} \pi = M\pi,$$

where, as usual, R is a 3×3 rotation matrix and $\vec{t}$ the translation vector of the displacement. So, the displacements of the plane which preserve the contact between the point $\tilde{p}$ and the plane π must satisfy the equation,

$$\tilde{p}^T M\pi = \tilde{p}^T \begin{pmatrix} R & 0 \\ -\vec{t}^{\,T} R & 1 \end{pmatrix} \pi = 0. \tag{3}$$

Substituting the Study parameters $a_0, \dots c_0, \dots c_3$, for the components of the 4×4 matrix M gives,

$$M = \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_0a_3) & 2(a_1a_3 + a_0a_2) & 0 \\ 2(a_1a_2 + a_0a_3) & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_0a_1) & 0 \\ 2(a_1a_3 - a_0a_2) & 2(a_2a_3 + a_0a_1) & a_0^2 - a_1^2 - a_2^2 + a_3^2 & 0 \\ \tau_x & \tau_y & \tau_z & \Delta \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} \tau_x &= 2(a_1c_0 - a_0c_1 - a_3c_2 + a_2c_3), \\ \tau_y &= 2(a_2c_0 + a_3c_1 - a_0c_2 - a_1c_3), \\ \tau_z &= 2(a_3c_0 - a_2c_1 + a_1c_2 - a_0c_3), \end{aligned} \tag{5}$$

and $\Delta = a_0^2 + a_1^2 + a_2^2 + a_3^2$, which has been included to make the equations homogeneous. Equation (3) can then be written as,

$$\mathbf{g}^T Q(p, \pi)\mathbf{g} = 0,$$

with $\mathbf{g}^T = (a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)$. These quadrics can be written as 8×8 symmetric matrices, see [6] for details on how this can be done.

As examples, suppose the fixed point is the origin, $\tilde{p}_0^T = (0, 0, 0, 1)$ and the plane is initially the xy-plane, $\pi_{xy}^T = (0, 0, 1, 0)$ or initially the xz-plane $\pi_{xz}^T = (0, 1, 0, 0)$, the quadrics for the possible displacements of these planes will be,

$$\mathbf{g}^T Q(\tilde{p}_0, \pi_{xy})\mathbf{g} = 2(a_3c_0 - a_2c_1 + a_1c_2 - a_0c_3)$$

and

$$\mathbf{g}^T Q(\tilde{p}_0, \pi_{xz})\mathbf{g} = 2(a_2c_0 + a_3c_1 - a_0c_2 - a_1c_3).$$

Now, it is tempting to think that the conditions for a single point to lie on a line can be thought of as a pair of point-plane constraints where the same point is constrained to a pair of planes intersecting along the given line. For example, the displacements which move the x-axis so that it remains in contact with the origin, lie in the intersection of the Study quadric with both $Q(\tilde{p}_0, \pi_{xy})$ and $Q(\tilde{p}_0, \pi_{xz})$. However, this is not the whole story.

Consider how this subspace of displacements could be parameterised. The dual quaternions which preserve the incidence of a point with a line form a $\mathbb{P}^3 \times \mathbb{P}^1$ Segre variety in the Study quadric. Suppose the point is initially located at the origin and is constrained to remain on the x-axis. Clearly any translation along the line will be such an element as will any rotation about the origin. Combining these gives a 4-dimensional set of dual quaternions that can be parameterised as,

$$g = (\mu_0 + \mu_1 i + \mu_2 j + \mu_3 k)(\lambda_0 + \varepsilon\lambda_1 i),$$

where μ_i and λ_j are arbitrary parameters. The parameters $(\lambda_0 : \lambda_1)$ can be thought of as homogeneous coordinates for a $\mathbb{P}^1$ and $(\mu_0 : \mu_1 : \mu_2 : \mu_3)$ for $\mathbb{P}^3$. The image in $\mathbb{P}^7$ is given in coordinates by $g = a + \varepsilon c$ with,

$$\begin{aligned} a &= \lambda_0\mu_0 + \lambda_0\mu_1 i + \lambda_0\mu_2 j + \lambda_0\mu_3 k, \\ c &= -\lambda_1\mu_1 + \lambda_1\mu_0 i + \lambda_1\mu_3 j - \lambda_1\mu_2 k. \end{aligned} \tag{6}$$

It is clear that any point-on-a-line can be transformed to this configuration using a suitable conjugation in the $SE(3)$. This Segre variety lies in several quadrics in $\mathbb{P}^7$, the equations of these quadrics can be found by asserting that the rank of the following matrix is 1 or less:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ c_1 & -c_0 & -c_3 & c_2 \end{pmatrix}.$$

This gives 6 quadrics,

$$\begin{aligned} Q_1 &= a_0c_0 + a_1c_1 = 0, & Q_4 &= a_1c_3 - a_2c_0 = 0, \\ Q_2 &= a_0c_3 + a_2c_1 = 0, & Q_5 &= a_1c_2 + a_3c_0 = 0, \\ Q_3 &= a_0c_2 - a_3c_1 = 0, & Q_6 &= a_2c_2 + a_3c_3 = 0. \end{aligned}$$

The variety will lie on any linear combination of these six. In particular, it is straightforward to check that the Study quadric and the quadrics $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xy})\mathbf{g}$ and $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xz})\mathbf{g}$, all lie in the linear system formed by these quadrics, the Study quadric is, $Q_1 + Q_6 = a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$, see (2). The other two are, $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xy})\mathbf{g} = 2(Q_5 - Q_2) = 0$ and $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xz})\mathbf{g} = -2(Q_3 + Q_4) = 0$.

The degree of such a Segre variety is $\binom{n+m}{n} = \binom{3+1}{1} = 4$. This suggests that there is another component to the intersection of the 3 quadrics. The intersection of the quadrics $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xy})\mathbf{g}$ and $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xz})\mathbf{g}$ is a five dimensional variety, that can be parameterised as,

$$\begin{aligned} a_0 &= \lambda_0\mu_0, & c_0 &= -\lambda_1\mu_1 + \lambda_2\mu_0, \\ a_1 &= \lambda_0\mu_1, & c_1 &= \lambda_1\mu_0 + \lambda_2\mu_1, \\ a_2 &= \lambda_0\mu_2, & c_2 &= \lambda_1\mu_3 + \lambda_2\mu_2, \\ a_3 &= \lambda_0\mu_3, & c_3 &= -\lambda_1\mu_2 + \lambda_2\mu_3. \end{aligned}$$

This is a linear projection of the Segre variety, $\mathbb{P}^2 \times \mathbb{P}^3$. Now, substituting this parameterisation into the Study quadric gives,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = \lambda_0\lambda_2(\mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^2)$$

Hence the intersection of all three quadrics consists of two 4-dimensional varieties. For the first $\lambda_2 = 0$ and the Segre variety discussed above is recovered. The second component arises when $(\mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^2) = 0$. All solutions to this condition will be complex and are thus not physically valid displacements. When $\lambda_0 = 0$ the result is the 3-plane A_∞, which lies in the both the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3$ and the complex residual variety.

The displacement variety described above can be realised with an SP dyad. That is, the end effector of a linkage consisting of a spherical and a prismatic joint will be able to adopt all displacements that maintain contact between a line parallel to the prismatic joint passing through the centre of the spherical joint. If the final link of this mechanism is rigidly attached to, say an RR linkage then, in general, the resulting closed-loop mechanism will have 4 assembly configurations. This is because the displacement variety of the RR mechanism is known to be the intersection of the Study quadric with a 3-plane. The general number of assembly configurations is the intersection of this variety with the $\mathbb{P}^3 \times \mathbb{P}^1$ Segre variety traced by the SP linkage. The Segre variety lies in the Study quadric so intersecting with the 3-plane gives 4 points; the degree of the Segre variety.

3 Two Ports

In robot surgery, several cannulas are used. Suppose the micro grippers at the ends of two such cannulas hold a rigid body, a needle perhaps. What rigid displacements can the body be subjected to now? In terms of mechanisms, this question is equivalent to asking for the coupler variety of a single loop mechanism consisting of two SP legs. The solution should be the intersection of two $\mathbb{P}^3 \times \mathbb{P}^1$ Segre varieties, but any such Segre variety will contain the 3-plane A_∞. Since we expect the variety to be 2-dimensional, the intersection cannot be a complete intersection. To get around this difficulty a birational map can be applied to the varieties. In [4] a birational map between the Study quadric and a variety defined by the standard homogeneous representation of $SE(3)$ was studied. Here, a map to the variety defined by the inverse-transpose to the standard representation will be used. This map is very similar to the one given in [4], in fact only few signs are changed. The map from the Study quadric is essentially given by equations (4) and (5), the non-zero elements of the matrix M are taken as homogeneous coordinates in a $\mathbb{P}^{12}$. The point of doing this is that the exceptional set of the map, the set on which the map is not defined, is just A_∞.

The inverse map is given by,

$$
\begin{aligned}
a_0 &= -2(\Delta + m_{11} + m_{22} + m_{33})\Delta,\\
a_1 &= 2(m_{23} - m_{32})\Delta,\\
a_2 &= 2(m_{31} - m_{13})\Delta,\\
a_3 &= 2(m_{12} - m_{21})\Delta,\\
c_0 &= -((m_{23} - m_{32})\tau_x + (m_{31} - m_{13})\tau_y + (m_{12} - m_{21})\tau_z),\\
c_1 &= (-(\Delta + m_{11} + m_{22} + m_{33})\tau_x - (m_{12} - m_{21})\tau_y + (m_{31} - m_{13})\tau_z),\\
c_2 &= ((m_{12} - m_{21})\tau_x - (\Delta + m_{11} + m_{22} + m_{33})\tau_y - (m_{23} - m_{32})\tau_z),\\
c_3 &= (-(m_{31} - m_{13})\tau_x + (m_{23} - m_{32})\tau_y - (\Delta + m_{11} + m_{22} + \rho_{33})\tau_z).
\end{aligned}
$$

As with the homogeneous representation, the image of the Study quadric in the $\mathbb{P}^{12}$ with coordinates m_{ij}, τ_k and Δ, can be shown to be the join of the 3-dimensional Veronese variety with a disjoint 2-plane. Call this variety Y, the degree of Y is thus 8, see [4]. The image of the two quadrics, $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xy})\mathbf{g} = 0$ and $\mathbf{g}^T Q(\tilde{p}_0, \pi_{xz})\mathbf{g} = 0$ are simply the hyperplanes $\tau_y = 0$ and $\tau_z = 0$. On the other hand the image of the Segre variety can be parameterised as,

$$
M_p = \begin{pmatrix}
\lambda_0(\mu_0^2 + \mu_1^2 - \mu_2^2 - \mu_3^2) & 2\lambda_0(\mu_1\mu_2 - \mu_0\mu_3) & 2\lambda_0(\mu_1\mu_3 + \mu_0\mu_2) & 0\\
2\lambda_0(\mu_1\mu_2 + \mu_0\mu_3) & \lambda_0(\mu_0^2 - \mu_1^2 + \mu_2^2 - \mu_3^2) & 2\lambda_0(\mu_2\mu_3 - \mu_0\mu_1) & 0\\
2\lambda_0(\mu_1\mu_3 - \mu_0\mu_2) & 2\lambda_0(\mu_2\mu_3 + \mu_0\mu_1) & \lambda_0(\mu_0^2 - \mu_1^2 - \mu_2^2 + \mu_3^2) & 0\\
-\lambda_1\Delta & 0 & 0 & \lambda_0\Delta
\end{pmatrix} \tag{7}
$$

with $\Delta = \mu_0^2+\mu_1^2+\mu_2^2+\mu_3^2$. This can be found either by multiplying an arbitrary translation in the x-direction by a general rotation about the origin or by substituting the parameterisation given in (6) into the map for the inverse-transpose representation (equations (4) and (5)) and cancelling the common factor λ_0. This can be thought of as the join of the Veronese variety with a single point, $\Delta = m_{ij} = \tau_y = \tau_z = 0$. Hence the image of the Segre variety has degree 8 and can thus be identified with the intersection of Y with the two hyperplanes.

Returning to the original problem, suppose we choose coordinates so that one point is located at the origin and the other at the point $\tilde{p}_a$, the x-axis will be aligned with the initial position of the first line and the second point will initially lie on the planes π_1 and π_2. Now, substitute the parameterisation for the first Segre variety (7), into the two equations for the second line-plane constraint,

$$
\tilde{p}_a^T M_p \pi_1 = 0, \quad \text{and} \quad \tilde{p}_a^T M_p \pi_2 = 0.
$$

The result will be a pair of equations linear in the "λ" variables and quadratic in the "μ" parameters. In fact it is clear that the equations will have the form, $-n_{ix}\Delta\lambda_1 + q_i\lambda_0 = 0$, where $i = 1, 2$; q_1 and q_2 are quadratic functions of the μ_js and n_{ix} is the x component of the normal vector to π_i. To get non-trivial solutions for the λs we must have that $q_1 n_{2x} - q_2 n_{1x} = 0$. This is a quadratic equation in the four homogeneous parameters $\mu_0, \ldots, \mu_3$, hence determines a 2-dimensional quadric in a $\mathbb{P}^3$. This is itself a Segre variety, $\mathbb{P}^1 \times \mathbb{P}^1$. So the μ_i can be written as linear functions of homogeneous parameters $\alpha_j\beta_k$ where $j,\ k = 0, 1$. That is, the μs are separately linear in the α and β variables. The solutions for the λs however, are quadratic in these variables, since, $\lambda_0 = n_{1x}\Delta$ and $\lambda_1 = q_1$. Substituting these results back into the parameterisation given in (6) will result in a parameterisation of a 2-dimensional variety in the Study quadric that is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ by a map that is separately cubic in the coordinates of each $\mathbb{P}^1$. Notice that, at any configuration the two lines can rotate about the axis joining the two fixed points $\tilde{p}_0$ and $\tilde{p}_a$.

Alternatively, if we use equation (7) to map this 2-dimensional variety into Y, the inverse-transpose representation of $SE(3)$, the image will have a parameterisation that is degree 4 in both the α and β parameters. Since the image variety is the intersection of Y with a pair of hyperplanes, it will have degree 8.

Finally, a parallel mechanism consisting of 3 SP linkages, that is three cannulas holding a rigid object, we would expect a finite number of assembly configurations. This is the intersection of the group variety Y with six hyperplanes. In general, the number of assemblies is then just the degree of Y, that is 8. However, if for example the lines are mutually parallel then a one degree-of-freedom motion is possible. This corresponds to the a linear dependency between the 6 hyperplanes.

4 Conclusions

The study of displacement varieties seems to be fundamental to the theory of mechanisms and linkages. Although the inspiration for this work comes from problems in surgical robotics it is hoped that there are wider applications of the results and techniques outlined. There are several other interesting problems relevant to robot surgery and other application. For example, how can general 6R robots be controlled to respect a geometrical constraint such as the line-on-point constraint studied here?

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