Some Remarks on the *RRR* Linkage

J.M. Selig

Faculty of Business
London South Bank University, U.K.

Introduction - Dual Quaternions/Study Quadric Reminder

Represent rigid displacements as dual quaternions,

$$(a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(b_0 + b_1 i + b_2 j + b_3 k)$$

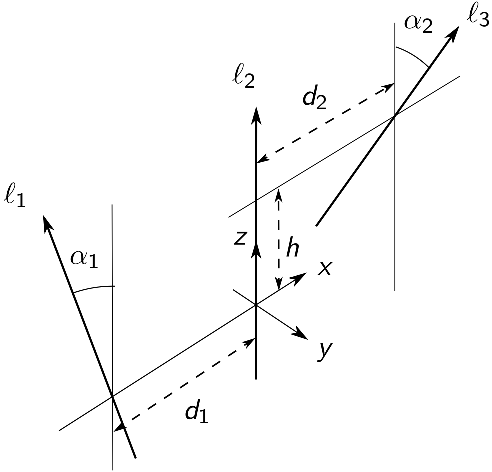
Think of a_0, $a_1, \ldots, b_3$ as homogeneous coordinates in a $\mathbb{P}^7$.

Rigid-displacements satisfy the equation,

$$a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$$

degree 2 equation, variety called the Study quadric. Points on the quadric in 1-to-1 correspondence with group elements except for those with $a_0 = a_1 = a_2 = a_3 = 0$.

The General 3R Linkage



Look at displacements generated by end-effector of general 3R linkage. Here ℓ_i is the ith joint axis.

Parametric Representation

Use dual quaternions to express rigid displacements. In general,

$$(a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(b_0 + b_1 i + b_2 j + b_3 k) = \\ (c_1 + s_1 \ell_1)(c_2 + s_2 \ell_2)(c_3 + s_1 \ell_3)$$

Here c_i, s_i are the parameters, can think of them as Cosine and Sine of joint half-angles. The joint axes are given by lines,

$$\ell = (\omega_x i + \omega_y j + \omega_z k) + \varepsilon(v_x i + v_y j + v_z k)$$

wth ω_i and v_j the components of the direction along the joint axis and its moment vector respectively.

Parametric Representation

Can write this in matrix form,

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = M \begin{pmatrix} c_1 c_2 c_3 \\ s_1 c_2 c_3 \\ c_1 s_2 c_3 \\ c_1 c_2 s_3 \\ s_1 s_2 s_3 \\ -c_1 s_2 s_3 \\ s_1 c_2 s_3 \\ -s_1 s_2 c_3 \end{pmatrix}$$

The matrix M has columns,

$$M = \left(1 \middle| \ell_1 \middle| \ell_2 \middle| \ell_3 \middle| \ell_1\ell_2\ell_3 \middle| -\ell_2\ell_3 \middle| \ell_1\ell_3 \middle| -\ell_1\ell_2\right)$$

The Matrix

Where,

$$1 = (1, 0, 0, 0, 0, 0, 0, 0)^T$$

and

$$\ell = (0, \omega_x, \omega_y, \omega_z, 0, v_x, v_y, v_z)^T$$

and so forth.

When M is non-singular this is just a projective transformation of the standard Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. That is just a (projective) change of coordinates.

Nine Quadrics - Implicit Representation

If we write,

$$X_0 = c_1c_2c_3, \quad X_1 = s_1c_2c_3, \quad X_2 = c_1s_2c_3, \quad X_3 = c_1c_2s_3,$$

$$Y_0 = s_1s_2s_3, \quad Y_1 = -c_1s_2s_3, \quad Y_2 = s_1c_2s_3, \quad Y_3 = -s_1s_2c_3.$$

then the standard Segre variety lies on 9 quadrics,

$$\begin{aligned} Q_1: \quad X_0Y_1 + X_2X_3 &= 0, \\ Q_2: \quad X_0Y_2 - X_1X_3 &= 0, \\ Q_3: \quad X_0Y_3 + X_2X_3 &= 0, \end{aligned} \qquad \begin{aligned} Q_4: \quad X_1X_0 + Y_2Y_3 &= 0, \\ Q_5: \quad X_2Y_0 - Y_1Y_3 &= 0, \\ Q_6: \quad X_3Y_0 + Y_1Y_2 &= 0. \end{aligned}$$

$$\begin{aligned} Q_7: \quad X_0Y_0 + X_1Y_1 &= 0, \\ Q_8: \quad X_0Y_0 - X_2Y_2 &= 0, \\ Q_9: \quad X_0Y_0 + X_3Y_3 &= 0. \end{aligned}$$

Could use M^{-1} to write 9 quadrics in the coordinates $a_0, a_1, \ldots, b_3$.

Application - Line Symmetric 6R Mechanism

Can the end-effector of such a $3R$ linkage undergo a line-symmetric motion?

If yes, then could join the end-effectors of two identical $3R$ linkages to get a mobile $6R$ closed-loop mechanism. The coupler bar will trace the line symmetric motion.

Line symmetric motions lie in the intersection of the Study quadric with certain 5-planes. Intersecting the Segre variety with a general $\mathbb{P}^5$ gives a curve in general. Hence the answer to the question above is: Yes. So line-symmetric $6R$ mechanisms are mobile.

Moreover, the degree of the curve will be 6, since it is well known that the Segre variety has degree six.

Genus of the Curve

In general this configuration curve must be elliptic (genus 1),

- A degree 6 curve in a $\mathbb{P}^7$ can be either rational or elliptic. Project from points until you get a cubic in $\mathbb{P}^2$ which is known to be rational or elliptic.
- Cannot be rational since it lies in 9 quadrics. A rational curve of degree six n $\mathbb{P}^7$ can only lie in 8 quadrics.

Many other easy consequences for closed loop mechanisms and 6R robots.

The Determinant of M

Using parameterisation above, can compute the determinant of M.

$$\begin{aligned}\det(M) = -h^4 \sin^4\alpha_1 \sin^4\alpha_2 \\ - 2h^2 \sin^2\alpha_1 \sin^2\alpha_2 (d_1^2 \sin^2\alpha_2 + d_2^2 \sin^2\alpha_1) \\ - (d_1^2 \sin^2\alpha_2 - d_2^2 \sin^2\alpha_1)^2.\end{aligned}$$

Careful analysis shows just 3 cases when $\det(M) = 0$,

Result due to Josef Schicho

The determinant of M is zero if and only if one of the following conditions hold,

- All joint axes are parallel, $\sin\alpha_1 = \sin\alpha_2 = 0$. Planar linkage.
- All joint axes are concurrent, $d_1 = d_2 = h = 0$. Spherical linkage.
- Design parameters satisfy, $h = 0$ and $\frac{d_1}{\sin\alpha_1} = \frac{d_2}{\sin\alpha_2}$. The Bennett conditions - call this $3R$ a Bennett linkage.

Variety Generated by the Bennett Linkage

Can show that the variety of displacements generated by the end-effector of such a Bennett linkage is the projection of the Segre variety from two point in the variety to a $\mathbb{P}^5$.

"Explains" mobility of the Bennett closed-loop $4R$ mechanism: Rotation about the 4th joint of the Bennett mechanism corresponds to a line in the Study quadric. The only lines in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are the generator lines - rotations about the first three joints. However, twisted cubic curves in the Segre variety which meet the two projection centres will project to lines.

In fact possible to show that variety generated by the Bennett linkage is the complete intersection of a 5-plane with the Study quadric and one other quadric. So a variety of degree 4.

Applications

The coupler of a line-symmetric Bennett-Bennett mechanism traces an elliptic quartic curve in general — the intersection of two quadrics in a $\mathbb{P}^3$.

Can find the general number of postures for a $6R$ serial robot with a Bennett linkage for its first three (or last 3) joints.

Bennett-$3R$ wrist: 4 postures.

Bennett-Bennett: 8 postures.

Bennett-General $3R$: 12 postures.

Several other easy results too.

Conclusions

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THANK YOU