Non-parametric Estimation of Copula Parameters: Testing for Time-Varying Correlation

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Abstract
The correlation structure of financial assets is a key input with regard to portfolio and risk management. In this paper, we propose a non-parametric estimation method for the time-varying copula parameter. This is achieved in two steps: first, displaying the marginal distributions of financial asset returns by applying the empirical distribution function; second, by implementing the local likelihood method to estimate the copula parameters. The method for obtaining the optimal bandwidth through a maximum pseudo likelihood function and a statistical test on whether the copula parameter is time-varying are also introduced. A simulation study is conducted to show that our method is superior to its contender. Finally, we verify the proposed estimation methodology and time-varying statistical test by analysing the dynamic linkages between the Shanghai, Shenzhen and Hong Kong stock markets.

Keywords: Time-varying Copula, Dynamic Dependence, Stock Returns, Kernel Estimate, Local Likelihood Estimation
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1. Introduction

Correlations between financial assets play a key role in portfolio and risk management. Understanding the time-varying nature of correlations allows for improved asset allocation, market timing and hedging decision-making. Further, following recent market volatility, the ability to model any time-variation within correlation has become increasingly important. Furthermore, the dependence structure between financial variables is often both non-linear and non-normal. As such, Li (2000) and Embrechts et al (2002) pointed out the limitations of using Pearson's linear correlation coefficient as a measure of the dependence and raised the issue of using a copula to describe non-linear relationships between financial assets.

A copula is a function that links marginal distributions of random variables to form a joint distribution. In other words, the joint distribution function can be written in terms of a copula and the marginal distribution functions. Thus, the copula contains all information on the dependence structure of random variables while the marginal distribution functions contain all information of the margins. That is, the copula provides a relatively straightforward way of modeling often non-linear and non-normal joint distributions that might otherwise only be examined through simulation approaches. This paper, therefore, seeks to present an advanced estimation method for the time-varying copula parameter. This method can be achieved in two steps: first, displaying the marginal distributions of the financial asset returns by applying the Empirical Distribution Function (EDF); second, implementing the local likelihood method to estimate the copula parameters. The method for obtaining the optimal bandwidth through maximum pseudo likelihood function and a statistical test on whether the copula parameter is time-varying are also introduced.

Parameter estimation is a primary task in the study of the copula. Joe (1997) discussed a computational method for the estimation of the copula parameters that acts as an alternative to the Maximum Likelihood (ML) estimation approach. This estimation approach is referred to as the method of Inference Functions for Margins (IFM) which displays the margins of dependent random variables by probability integral transformation functions. However, Chen and Fan (2006b) noted that the dependence parameters could be affected by a possibly misspecified marginal distribution of the standardised innovations. Thus, they suggested using the Canonical Maximum Likelihood (CML) estimation approach, where the margins are specified by the empirical distribution function. Kim et al., (2007) compared the ML and IFM methods with the semi-parametric method proposed by Genest et al., (1995). Similar to Chen and Fan (2006a), they found that ML and IFM methods are non-robust against misspecification of the marginal distributions and the semi-parametric method, where the margins are estimated non-parametrically by their empirical distributions, performs better than the ML and IFM methods.

To avoid such misspecification in this paper, we also specify the margins non-parametrically by the empirical distribution function. Thereby, reducing the deviation caused by any misspecification of the marginal distribution. To improve the computational efficiency we use a two-step estimation method: first, displaying the marginal distributions by applying the empirical distribution function, then implementing the local likelihood method in parameter estimation, where the dependence structure is modelled by a deterministic function of time.

Capturing the time varying nature of the copula parameters has received increased attention in the last decade, not least because it has become increasingly recognised that the dependence structure of financial assets is time-varying. Patton (2006a,b) assumed that current dependence is reliant on previous dependence and the historical average differences of the cumulative probabilities of the two series (see, also Patton, 2009). Other studies for the estimation of time-varying copulas include Rodriguez (2007), who proposed a time-varying mixture copula where the weight of each copula and their marginal distributions follow a two stage switching process. Jondeau and Rockinger (2006) assumed that the dependence is a function of historical realisations and a deterministic function of time.

Further work, modeling the copula parameter, directly or indirectly, as a function of time includes Bartram et al (2007) and Ane et al (2008). However, due to the uncertainty and complexity of the correlation structure between financial assets, there is a high degree of subjectivity in assuming the nature of the evolution function of the time-varying copula parameters. Thus, in this paper, instead of assuming the time-varying copula parameter α follows a specific evolution function, we estimate α non-parametrically as an unknown function of time $\alpha(t)$ by the local likelihood method, thereby reducing the bias caused by any misspecification.

This paper is closely related to the work of Hafner and Reznikova (2010), but differs in a variety of ways: First, we apply the CML method where the margins are captured by the empirical distribution function, while Hafner and Reznikova (2010) specify the margins by parametric GARCH-type processes. As noted above the CML method offers an advantage as a consistent estimator of the true cumulative distribution function. Although both approaches apply the local likelihood method for copula parameter, second, we obtain the optimal bandwidth through a maximum pseudo likelihood function which is computationally convenient. A further statistical test on whether the copula parameter is time-varying is also introduced here. Finally, we use two simulation studies to show that our method is not only feasible but also superior to other benchmark methods in the literature.

In order to further verify the proposed estimation methodology and the time-varying hypothesis test, we investigate the dynamic linkages between Shanghai, Shenzhen and Hong Kong stock markets. We find that market integration between mainland China and regional developed markets, such as Hong Kong, has significantly increased since the reform of non-tradable shares in 2005. While the financial crisis hit the Hong Kong market badly and caused a decrease in correlation between mainland China and the Hong Kong stock market, it did not have a huge impact on the dependence structure between stock markets in mainland China. Finally, we find the correlation for Shanghai/Hong Kong and Shenzhen/Hong Kong significantly increased in early 2009, when the Chinese stock market started its bear market. This finding is consistent with Longin and Solnik (2001), which suggests that stock markets are more correlated in bear markets. The increased market integration suggests that there will be less portfolio diversification opportunities. However, at the end of our sample period, Shenzhen/Hong Kong became less dependent after 2010, while this is not observed for Shanghai/Hong Kong.

2. Estimating the time-varying copula parameter

Assume the return series of financial assets A and B have a joint distribution function given by $H(r_A, r_B)$. The cumulative distribution functions for r_A and r_B are defined as $F_A(r_A)$ and $F_B(r_B)$ respectively. By Sklar's Theorem, there exists a copula function C_α that links the two marginal distributions (Nelsen, 1998):

$$C_\alpha(u_{t,} v_t) = C_\alpha(F_A(r_A^t), F_B(r_B^t)) = H(r_A^t, r_B^t) \tag{1}$$

where $u_{t,} v_t$ are marginal distributions and $u, v \in [0,1]$. In the following section, we will use a two-step method for parameter estimation of the time-varying copula. In step 1, we display the marginal distribution by the empirical transformation function. Chen and Fan (2006b) found that if incorrect cumulative distribution functions are used for marginal distributions of, say, $\{x_t\}$ and $\{y_t\}$, large deviations might occur in the estimation process. They suggested using the empirical measure for finite samples as the empirical distribution function is a consistent estimator of the overall distribution function. Therefore, we employ the empirical distribution function of $\{x_t\}$ and $\{y_t\}$ as an estimator of their overall distribution function and let:

$$\hat{u}_t = \hat{F}_X(x_t) = \frac{1}{T}\sum_{j=1}^{T} I(x_j \leq x_t), \hat{v}_t = \hat{F}_Y(y_t) = \frac{1}{T}\sum_{j=1}^{T} I(y_j \leq y_t) \tag{2}$$

where $I(\Theta)$ is the indicator of event Θ, $\hat{F}_X(\cdot)$, $\hat{F}_Y(\cdot)$ are the cumulative distribution functions of X and Y. If $\{(x_t, y_t)\}_{t=1}^{T}$ are the return series of financial assets A and B then the return series $\{(r_A^t, r_B^t)\}_{t=1}^{T}$ can be transformed to $\{(\hat{u}_t, \hat{v}_t)\}_{t=1}^{T}$ by the above function, where $\hat{u}_t = \hat{F}_A(r_A^t)$, $\hat{v}_t = \hat{F}_B(r_B^t)$, are the margins and $\hat{F}_A(\cdot)$, $\hat{F}_B(\cdot)$ are the cumulative distribution functions of the return series r_A and r_B.

In step 2, we estimate the time-varying copula parameter by applying the local likelihood estimation method. Although the copula parameter is time-varying, it can always be assumed as a deterministic function of time within a small time interval. Hafner and Reznikova (2010) use the local constant method to approximate the copula parameter. In this paper, we assume the copula parameter to be piecewise linear and approximate it with a local linear function. Using the local linear method not only simplifies the computational process, but also generates estimates with smaller MISEs (Minimum Integrated Squared Errors) than local constant estimates and overcomes the boundary effects (Fan and Gijbels, 1996).

In the following section, we assume that the second derivative of the time varying parameter α_t exists and is continuous. For a given time, t_0, we locally approximate α_t by the following procedure $\alpha_t \approx \beta_1 + \beta_2(t - t_0)$, where $t \in [t_0 - h, t_0 + h]$. Therefore, the local pseudo likelihood function at time t_0 can be written as:

$$L(\beta_1, \beta_2; h, t_0) = \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t; \beta_1, \beta_2)\, K_h(t - t_0) \tag{3}$$

where $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$, $h > 0$ is the smoothing parameter, and $K_h(\cdot) = \frac{1}{h} K(\cdot/h)$. $K(\cdot)$ is a two-side kernel estimation function, since the estimation of α_{t_0} is based on time interval $[t_0 - h, t_0 + h]$. In the section below, we use the two-side Epanechnoikov kernel function in our estimation. The Epanechnikov kernel has a smooth estimated density function and is generally regarded as a good kernel function, see Fan and Gijbels (1996). The kernel function can be expressed as: $K(u) = 0.75(1 - u^2)I_{-1 \le u \le 1}$. Thus, with a fixed smoothing parameter, h, we maximise the likelihood function over parameters β_1, β_2, to find the estimated value for the copula parameter at time t_0: $\hat{\alpha}_{t_0} = \hat{\beta}_1$. A further reason to use the Epanechnoikov kernel lies in its popularity since it is one of the most commonly used kernel functions (the two most commonly used kernel functions are Gauss kernel and Epanechnoikov kernel). Notwithstanding this, the choice of the kernel is less important in non-parametric modelling, as it has little effect on the final estimates. However, the choice of the smoothing parameter is crucial and therefore will be discussed separately in the next section. It is also worth mentioning that the above discussion is restricted to a one parameter copula function. If the copula function has more than one parameter, for example, the student t copula or the SJC copula, the simulation process can be done by taking the first order Taylor approximation for both parameters. In the case of two copula parameters, we locally approximate the parameters by the following functions:

$$\rho_t \approx \beta_1 + \beta_2(t - t_0), Dof_t \approx \beta_3 + \beta_4(t - t_0),$$

where $t \in [t_0 - h, t_0 + h]$. Then, equation (3) can be rewritten as:

$$L(\beta_1, \beta_2, \beta_3, \beta_4; h, t_0) = \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t; \beta_1, \beta_2, \beta_3, \beta_4)\, K_h(t - t_0) \tag{4}$$

The corresponding copula parameters can be estimated by maximising the above function.

The marginal distributions are assumed to be continuous, but without any functional form. Therefore, using the ECDF for margins is likely to be more consistent than methods such as Patton (2006a, b) and Hafner and Reznikova (2010), which requires the marginal distributions to be specified up to a finite number of unknown parameters, and are likely to be inconsistent even if just one marginal distribution is misspecified. Further, since we estimate each margin non-parametrically, where the sample size is moderate to large, the estimates from the ECDF-LML (empirical cumulative distribution function – local maximum likelihood) are asymptotically normal.[1] The above two-step ECDF-LML method is estimated level by level, plugging in parameters from previous levels at each step, which increases the computational speed considerably, but reduces its asymptotic efficiency.

3. Bandwidth selection

Bandwidth selection plays an important role in non-parametric density estimation. The optimal bandwidth is often obtained by minimising the Mean Integrated Square Error (MISE) (see, for example, Fan et al, 1998; Hafner and Reznikova, 2010). Other popular methods for bandwidth selection include: LSCV, MLCV and GCV.[2] In this paper, the optimal bandwidth is obtained by the following procedure: first, we limit our bandwidth selection to a possible set $\{h_1, h_2, \dots, h_m\}$; second, for each $h_i, i = 1,2, \dots, m$, we obtain the pseudo log-likelihood function:

$$\sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t, \hat{\alpha}(t,h); h = h_i)$$

[1] The asymptotically normal property is described in Genest et al (1995), Tsukahara (2005) and Hafner and Reznikova (2010) for ECDF and local likelihood method separately.
[2] LSCV: Least-Square Cross-Validation; MLCV: Maximum Likelihood Cross-Validation; GCV: Generalised Cross-Validation.

where $\hat{\alpha}(t,h)$ is obtained from equation (3) by maximising the likelihood function over parameters β_1, β_2:

$$(\hat{\beta}_1, \hat{\beta}_2) = \mathrm{argmax}_{\beta_1,\beta_2} \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t; \beta_1, \beta_2)\, K_h(t - t_0) \tag{5}$$

Third, the optimal bandwidth is the one that maximise the pseudo log-likelihood function:

$$\hat{h}_{opt} = arg\,\max_h \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t, \hat{\alpha}(t,h); h) \tag{6}$$

4. Time-varying hypothesis testing

A large amount of literature states that the dependence between financial asset returns is time-varying, such as Longin and Solnik (1995), Engle (2002), Rodriguez (2007), Patton (2006a,b). However, none of these have tested the time-varying nature of the dependence structure. In this paper, we perform a generalised maximum pseudo-likelihood ratio test for the time-varying nature of the copula parameter. The theoretical background of this method is the so-called Wilks phenomenon introduced in Fan et al (2001). Fan et al (2001) proposed a generalised likelihood statistic to overcome the failure of applying a maximum likelihood ratio test on most non-parametric test settings. Based on the local linear fitting and some sieve methods, they show that the genreralised likelihood test achieves the optimal minimax rate for hypothesis testing and, with proper choice of the smoothing parameters this likelihood test is indeed general and powerful for most non-parametric settings. This further supports for the use of the generalised likelihood method in our paper.

The test is based upon the following hypothesis:

$$H_0: \alpha_t = \alpha_0 = const; H_1: \alpha_t \neq \alpha_0 \neq const$$

the pseudo log-likelihood function for H_1 is:

$$\lambda(H_1) = \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t, \hat{\alpha}_t; \hat{h}_{opt}) \tag{7}$$

where $\hat{h}_{opt}$ is the optimal bandwidth determined by equation (5), $\hat{\alpha}_t$ is the time-varying parameter under optimal bandwidth $\hat{h}_{opt}$. Similarly, we have the pseudo log-likelihood function for H_0:

$$\lambda(H_0) = \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t, \hat{\alpha}_0) \tag{8}$$

where

$$\hat{\alpha}_0 = arg\max_{\alpha} \sum_{t=1}^{T} logc(\hat{u}_t, \hat{v}_t, \alpha) \tag{9}$$

The pseudo-likelihood ratio test is expressed as:

$$\lambda = 2(\lambda(H_1) - \lambda(H_0)) \tag{10}$$

where the distribution of the test statistic λ is obtained through a bootstrap method, as described in Cai et al (2000) and Fan et al (2003). Under the null hypothesis, the bootstrap is implemented by computing critical values or P-values for λ as described in equation (6) as follows:

Step 1: Simulate a sample of observed values $\{(u_t^*, v_t^*)\}_{t=1}^{T}$ from a specified copula with given constant parameter $\hat{\alpha}_0$ in equation (8)

Step 2: Estimate the time varying copula parameter by applying the ECDF-LMLE method from the bootstrap sample $\{(u_t^*, v_t^*)\}_{t=1}^{T}$

Step 3: Computing λ^* using equation (9) on the bootstrap sample $\{(u_t^*, v_t^*)\}_{t=1}^{T}$

Step 4: Repeat Step1 to 3 for $N = 1000$ times and compute the bootstrap P-values.

5. Simulation

In this section, we implement a Monte-Carlo simulation to evaluate the performance of the ECDF-LML method by comparing it with a benchmark estimation method found in literature. In particular, we consider the method described in Patton (2006a,b) as the benchmark method for the time-varying copula parameters. The aim, therefore, is to demonstrate that the method here in estimating time-varying copula parameters is superior to the benchmark method. Two scenarios are conducted to investigate the performance of ECDF-LML estimates. The first scenario compares the ECDF-LML estimates directly to the estimates from Patton's method in terms of finite sample bias and variance, in order to demonstrate the robustness of our results. In the second scenario, a Monte-Carlo simulation study is carried out to compare the estimates obtained in the first scenario for different sample sizes in term of the generated Mean Squared Errors (MSEs) and Mean Absolute Relative Errors (MAREs).

Scenario 1 Robustness Check

The following example compares the ECDF-LML estimates directly to estimates generated by Patton's method. The copulas that have been tested include the Gaussian, student-t, and the SJC copula. In this section, we only report the results for the Gaussian copula as the other copulas delivers similar results, nevertheless, the simulation results for the other copulas are available upon request.

Following Nelson (2006), we design a three-step simulation process to generate the true and fitted values of the time-varying copula parameters. The true value refers to the copula parameter generated by Patton (2006)'s method and the fitted value is the ECDF-LML estimates. The three-step simulation process is described as follows:

First, the true value is estimated though the following function as described in Patton (2006), with the initial value $\rho_0 = 0.5$, and the initial values of (u_t, v_t) randomly generated from a uniform distribution[1].

$$\rho_t = \Lambda(\omega - \frac{\alpha}{10}\sum_{i=1}^{10}|u_{t-i} - v_{t-i}| + \beta\rho_{t-1}) \tag{11}$$

where $\Lambda(x) = \frac{1-e^{-x}}{1+e^{-x}}$ is the logistic transformation function which limits the range of $\rho_t \in [-1,1]$. Using a bootstrap method, we insert the value of ρ_{t-1} from the previous step for ρ_t and the value (u_{t-1}, v_{t-1}) is generated from a known bivariate distribution which is constructed by a Gaussian copula with parameter ρ_{t-1}.[3] Applying the bootstrap $N = 1000$ times we obtain the true value of the time-varying copula parameters$(\hat{\rho}_t)_{\mathrm{t=1}}^{1000}$. We remove the first 200 estimates to avoid any impact from the initial value, ρ_0 , and initial value of (u_t, v_t), thus we have: $(\hat{\rho}_t)_{\mathrm{t=201}}^{1000}$.

In the second step, we also obtain a bootstrap sample $\{(u_t^*, v_t^*)\}_{t=201}^{1000}$. This sample data is then used to calculate the ECDF-LML estimates $(\hat{\rho}_t)_{\mathrm{t=201}}^{1000}$. Finally, by repeating the previous steps 1000 times, we obtain 1000 estimated values for each $(\hat{\rho}_t)_{\mathrm{t=201}}^{1000}$. The fitted value is then calculated by taking the average of the 1000 estimated values for each $(\hat{\rho}_t)_{\mathrm{t=201}}^{1000}$.

Table 1 shows the descriptive statistic of the time-varying copula parameters. We find that the true value and the fitted value are very close on first quartile, mid value, average value, and third quartile. The purpose of doing this is to demonstrate the feasibility and robustness of the above proposed method.

We also test the effect of changing sample size. Three different sample sizes are chosen. As the sample size increase, we would expect to see the estimator's variance decrease. These results are presented in Table 2, where we list the deviations and the mean square errors for three different sample sizes at time point $t = 0.1T, 0.25T, 0.5T, 0.75T, 0.9T$, where T is the sample size. The parameters in equation (11) are defined as $\omega = 1.6$, $\beta = 0.8$ and $\alpha = -3.6$. We find that at each time point $t = 0.1T, 0.25T, 0.5T, 0.75T, 0.9T$, the MSE of the time-varying parameters are very small, and indeed decreases with the increase in sample size. For $T = 500$,

3 The method for simulating dependent random variables using copula can be found in Nelson (2006).

the finite sample bias and MSEs of the correlation parameter ρ are not discouragingly high. However, as T decreased to 100, it appears that none of the parameters are well estimated.

We also fit three pairs of parameters to the evolution function, which are $\omega = 1.2$, $\beta = 0.3$ and $\alpha = -2.6$; $\omega = 1$, $\beta = 0.7$ and $\alpha = -2$; $\omega = 0.219$, $\beta = 0.695$ and $\alpha = 0.627$, and fix the sample size T at 1000. The obtained the deviations and MSEs at time point $t = 0.1T, 0.25T, 0.5T, 0.75T, 0.9T$ are presented in Table 3. As can be seen, the values of the MSE are small in all cases, which indicate that the ECDF-LML estimation method is relatively stable. The true values and the fitted values of the time-varying parameters are further illustrated in Figure 1. The errors and the mean square errors are presented in Figure 2. All of these results suggest that our simulation method is feasible and robustness following Patton's approach.

Finally, we provide the hypothesis test for the time-varying copula parameters. The statistic should be 0.5805 if the parameter is constant over time. We use a bootstrap to repeat the simulation 500 times, and calculate the corresponding P value as 0.004. This suggests the rejection of the null hypothesis. Therefore, the correlation is time-varying.

Scenario 2 Test for Superiority

In this particular example we use a student t copula, other copulas give similar results. We first consider the following two processes for the time-varying dependence parameter.

Deterministic case:

$$\rho_t = 0.5 + 0.4\cos(2\pi t/200)$$

Stochastic case:

$$\text{AR(1)}: \rho_t = (\exp(2\lambda_t) - 1)/(\exp(2\lambda_t) + 1)$$

$$\lambda_t = 0.02 + 0.97\lambda_{t-1} + 0.1\varepsilon_t$$

$$\varepsilon_t \sim N(0,1)$$

In each process, we calculate the true value of ρ_t, using three different sample size ($T = 300, 500, 800$). Using the algorithm introduced in Nelson (2006), we randomly draw a pair of uniform variables(u_t, v_t) for each ρ_t from a predefined student t copula with a fixed degrees of freedom, six. Due to the extremely high complexity involved, only 300 Monte Carlo replications are performed. The student t copula is defined as follows:

$$C_t^{v,\rho}(x,y) = \int_{-\infty}^{t_v^{-1}(x)} \int_{-\infty}^{t_v^{-1}(y)} \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \{1 + \frac{(s^2 - 2\rho st + t^2)}{v(1-\rho^2)}\}^{-(v+2)/2} ds dt \tag{12}$$

where ρ is the correlation coefficient, and $t_v^{-1}(u)$ is the inverse of the standard t distribution with degree of freedom v. The time-varying parameter ρ_t described in Patton (2006) has the following expression

Through estimation, we have $\{(U_t, V_t)_{t=1}^T\}_1^N$, $N = 300$ and is the number of Monte Carlo replications. In each Monte Carlo replication, we calculate the time varying copula parameter by ECDF-LML and Patton's methods respectively.

The time varying correlation coefficient in Patton (2006) is described as follows:

$$\rho_t = \Lambda(w + \beta_1 \rho_{t-1} + \beta_2 \frac{1}{10} \sum_{i=1}^{10} \Phi^{-1}(u_{t-i}) \Phi^{-1}(v_{t-i})) \tag{14}$$

where $\Lambda(x) = \frac{1-e^{-x}}{1+e^{-x}}$ is the logistic transformation function to ensure $\rho_t \in [-1,1]$.

The performance is evaluated by the two measures:

The Mean Squared Error: $MSE = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{T} \sum_{t=1}^{T} (\hat{\rho}_t^n - \rho_t^0)^2$

The Mean Absolute Relative Error: $MARE = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{T} \sum_{t=1}^{T} |(\hat{\rho}_t^n - \rho_t^0)/\rho_t^0|$

where N is the number of Monte Carlo replications, T is the sample size and $\hat{\rho}_t^n$ and ρ_t^0 denote the estimated and the true copula parameter in the n's replication.

The results are summarised in Table 4. For each sample size, we define two series of true values for copula parameters which follow deterministic and stochastic functions respectively. We compare the estimated parameters by means of MSE and MARE. As can be seen, smaller MSEs and MAREs are observed using ECDF-LML method, reflecting its higher asymptotic efficiency. Furthermore, we observe decreased MSEs and MAREs with increased sample sizes using our method in the deterministic case, which demonstrate that the ECDF-LML method produces consistent estimates. However, this is not observable using Patton's method. Indeed, for the copula parameters, the ECDF-LML appears to suffer slightly less under the reduced sample size than its competitor. Another way to compare the two approaches is through examining the time-series plot for the copula parameters and the boxplot for the estimated errors. Figure 3 plots the time-series for true parameters and the copula parameter estimates using ECDF-LML and Patton's methods. We only plot the case for the sample size $T = 300$. Other sample sizes give similar results. As can be seen, the ECDF-LML produce a set of fitted values that are closer to the true values than the method of Patton. Figure 4 presents the boxplots for the estimated errors at time $t =$ 10%, 25%, 50%, 75% and 90% of the total sample. Again, we only present the case with sample size $T = 300$ as other sample sizes produce similar results. All the time-varying parameters are generated by student t copula with correlations following deterministic and stochastic functions. The number of Monte Carlo replications is 300. In each case, the first figure is calculated form ECDF-LML method while the second figure is by Patton's method. As can be seen, the second quartile (the median) of the estimated errors of our method is closer to 0 than the alternative approach. Therefore, our method produces more reliable estimates than Patton's. Both figures demonstrate that the performance of the ECDF-LML estimates are satisfactory compared to its contender.

6. Empirical application on three main Chinese stock markets

Chinese stock markets exemplify many characteristics of emerging markets, which differentiates them from developed financial market. Furthermore, an examination of Chinese financial markets is worthwhile in order to understand the applicability or otherwise of established financial theory to China.

In this section, we investigated the dynamic dependence between the stock index returns on the Shanghai, Shenzhen and Hong Kong stock markets. This will allow us to discuss how these markets respond to information and whether they were affected by the same information. Moreover, this empirical exercise is used to verify the proposed estimation and testing methodology. Weekly stock market indices are obtained from the Wind Database, with data coverage from 2003.1.3 to 2012.6.28. The indices are the Shanghai Stock Exchange Composite Index (SH), the Shenzhen Stock Exchange Composite Index (SZ) and the Hong Kong Hang Seng Index (HS). We have in total 548 observations after removing any non-matching data due to different trading dates.

6.1. Descriptive statistics

Table 5 displays some basic statistic information for the three return series. As can be seen, the average return of the Shenzhen stock market is the highest amongst the three, followed by the Hong Kong stock market. The stock markets in mainland china are slightly more volatile than the Hong Kong stock market, and have negative skewness, which indicate the existence of tail risk; while for Hong Kong the value is positive. Using the Jarque-Bera test we also find significant evidence of departure from normality.

6.2. Empirical analysis and results

Due to the non-normal, notably, fat tail behavior observed for the three stock returns, we use the student-t copula to examine the dependence between these three major stock markets in China. Table 5 shows the descriptive statistics of the time-varying correlation coefficients as modelled by the student-t copula. We find the Shanghai and Shenzhen stock markets are highly correlated with 75 percent of the correlation coefficients higher than 0.84. This demonstrates a very close relationship between the two markets. In contrast, Shanghai and Hong Kong (SH/HS) and Shenzhen and Hong Kong (SZ/HS) have lower constant correlation parameters, indicating a weaker integration between mainland China and the regional developed market. Nonetheless, for the two pairs, 25 percent of the time-varying correlation coefficients are higher than 0.47, indicating that a high degree of dependence is often observed. The P value of the time-varying test suggests that the correlation coefficients are time-varying.

The dynamic dependences between the three stock markets are presented in Figures 5, 6 and 7 where the horizontal line is the constant copula parameter. We find that the correlation between these markets varies significantly over time. For the SH/SZ correlation, while these two markets are distinct with dual listing not allowed, there exists a consistently higher correlation between them, with the average correlation coefficient above 0.9. Compared to the SH/SZ pair, the SH/HS and SZ/HS pairs are less correlated but that correlation exhibits greater fluctuation, with the dynamics appearing to respond to financial events. We also find that time-varying correlations between SH/HS and SZ/HZ have very similar patterns. Both fluctuate at a lower level from 2003 to 2005 which indicates a weaker integration between stock markets in mainland China and the Hong Kong stock market. Moreover, evidence of significant increases in correlation after 2005 is observed between these stock markets, which may point to evidence of financial contagion.

As can be seen from Figures 6 and 7, there are several peaks and sharp increases during the sample period. There was a significant rise in correlation in 2005. This can be considered as a direct response to the 2005 non-tradable share (NTS) reform, which leads to an enhanced relation between China and regional developed markets. However, as reflected by Figure 5, the reform had only a very limited impact on the dependence structure between the stock markets in mainland China. Following this initial increase, the correlation between mainland China and Hong Kong continue to fluctuate at a higher level, with average correlation at about 0.6.

The correlations for SH/HS and SZ/HS start to fall during the subprime crisis and fluctuate at a relatively lower level. This is perhaps surprising as much of the literature reports that the cross market correlation will increase significantly after a shock. This fall of correlation may be due to the different movements these stock markets experienced in response to the 2007 financial crisis. As the Hong Kong stock market is regarded as one of the more mature financial centers and plays an important role in the regional economy, it experienced a larger response to the financial crisis. On the other hand, the crisis did not seem to have a huge impact on the Shanghai and Shenzhen stock markets. Although the Chinese economy is closely linked to the rest of the world, there is only limited stock trading open to foreign investors through the trading of B shares. This leads to a low integration between the Chinese stock markets and the world economy. For the financial crisis that started in July 2007, we do not observe a significant change in the dependence structure for SH/SZ pair, which indicates that the markets of mainland China were not affected in the same way as Western stock markets. This, of course, has important practical implication for investors as they can reduce total risk by diversifying into Chinese stocks.

The final significant increase was observed in early 2009, when the Chinese stock market started its bear market. This finding is consistent with Longin and Solnik (2001), where they found that stock markets are more correlated in bear markets. This increased market integration suggests that there will be less portfolio diversification opportunities between these markets. Our results also suggest that the correlation at the end of the sample period is higher than the earlier 2000s which point to the evidence of further market integration towards the end of the sample period. However, the difference is, Shenzhen and Hong Kong stock markets became less dependent after 2010 for a short period, while the correlation between Shanghai and Hong Kong continuously fluctuates at a higher level.

7. Summary and Conclusion

The main purpose of this paper is to propose a non-parametric method to estimate the time-varying parameters on copula. In particular, this is achieved in two steps by first, displaying the marginal distributions of financial asset returns by their empirical distribution function and second, implementing the local likelihood method to estimate the copula parameters. The method for obtaining the optimal bandwidth through a maximum pseudo likelihood function and a statistical test on whether the copula parameter is time-varying are also introduced. Furthermore, we use two simulation studies to show that the proposed method is feasible and superior to its nature competitor. Finally, we use the proposed method to investigate the relationships between Shanghai, Shenzhen and Hong Kong stock markets.

Simulation results support our proposed estimation method, with only small MSEs obtained across a range of sample sizes and parameter values. With respect to the empirical exercise, we report that market integration between mainland China and the regionally developed market of Hong Kong has significantly increased since the reform of non-tradable shares in 2005. While the financial crisis hit the Hong Kong market badly and caused a decrease in correlation between mainland China and the Hong Kong stock market, it did not have a huge impact on stock markets in mainland China. Finally, we found the correlation for Shanghai/Hong Kong and Shenzhen/Hong Kong significantly increased in early 2009, when the Chinese stock market started its bear market. This finding is consistent with Longin and Solnik (2001), which suggests that stock markets are more correlated in bear markets. The increased market integration suggests that there will be less portfolio diversification opportunities. However, at the end of our sample period, SZ/HS became less dependent after 2010 for a short period, while this is not observed for SH/HS pair.

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Table 1 Descriptive Statistics of the Time-varying Gaussian Copula Parameters

Statistics	Min	First Quartile	Mid	Ave	Third Quartile	Max
True value	0.0098	0.5025	0.5997	0.5683	0.6657	0.7450
Fitted value	0.1595	0.5014	0.5923	0.5598	0.6502	0.7235

Note: results are based on parametric setting $\omega = 1.6$, $\beta = 0.8$ *and* $\alpha = -3.6$

Table 2 Simulation Results by Different Sample Size

	t= $0.1T$		$t = 0.25T$		$t = 0.5T$		$t = 0.75T$		$t = 0.9T$	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
$T = 100$	-0.025	0.023	-0.012	0.053	0.016	0.079	0.042	0.047	-0.007	0.043
$T = 300$	-0.002	0.024	-0.101	0.049	-0.01	0.027	-0.058	0.047	-0.001	0.066
$T = 500$	0.015	0.023	0.001	0.05	0.004	0.024	-0.008	0.042	-0.005	0.046

Note: results are based on parametric setting $\omega = 1.6$, $\beta = 0.8$ *and* $\alpha = -3.6$

Table 3 Simulation Results by Different Parameter Settings

	$\omega = 1.2, \beta = 0.3$ $\alpha = -2.6$		$\omega = 1, \beta = 0.7$ $\alpha = -2$		$\omega = 0.219, \beta = 0.695$ $\alpha = 0.627$	
	Bias	MSE	Bias	MSE	Bias	MSE
t= $0.1T$	-0.0257	0.095	-0.027	0.066	-0.027	0.083
$t = 0.25T$	-0.028	0.053	-0.011	0.065	-0.05	0.041
$t = 0.5T$	-0.012	0.071	0.003	0.078	0.044	0.05
$t = 0.75T$	-0.019	0.073	-0.003	0.079	0.032	0.038
$t = 0.9T$	-0.02	0.089	-0.023	0.087	-0.021	0.017

Note: Sample size T *is fixed at 1000.*

Table 4 MSE and MARE for estimated time varying student t copula parameter ρ

		Deterministic Case		Stochastic Case	
		MSE	MARE	MSE	MARE
$T = 300$	ECDF-LML	0.030	0.034	0.029	0.030
	Patton	0.051	0.051	0.032	0.032
$T = 500$	ECDF-LML	0.018	0.010	0.031	0.031
	Patton	0.052	0.101	0.065	0.124
$T = 800$	ECDF-LML	0.016	0.024	0.027	0.056
	Patton	0.054	0.137	0.049	0.132

Note: For each sample size, we consider two functions for the true copula parameter: deterministic and Stochastic. The number of Monte Carlo replication is 300. The estimates are compared with the true copula parameter by means of MSE and MARE.

Table 5 Descriptive Statistics of the Three Stock Returns

	Mean	Median	Maximum	Minimum	Std. Dev.	Skewness	Kurtosis
Shanghai	0.128	0.000	14.948	-13.863	3.533	0.233	4.951
Shenzhen	0.227	0.120	16.817	-15.343	4.035	-0.058	4.535
Hang Seng	0.193	0.467	12.433	-16.319	3.118	-0.126	5.832

Table 6 Descriptive Statistics of the Correlation Coefficient as Modeled by the Student *t* Copula

	Time-varying Parameter						Constant Parameter	TV Test	
	Min	First Quartile	Mid	Avg	Third Quartile	Max		Test Statistics	P-value
SH-SZ	0.751	0.838	0.930	0.905	0.964	0.988	0.908	-62.45	0
SH-HS	-0.116	0.202	0.452	0.403	0.600	0.771	0.362	-25.46	0
SZ-HS	-0.153	0.115	0.299	0.299	0.476	0.725	0.260	-30.21	0

Figure 1. Time-varying Parameters from Simulated Data

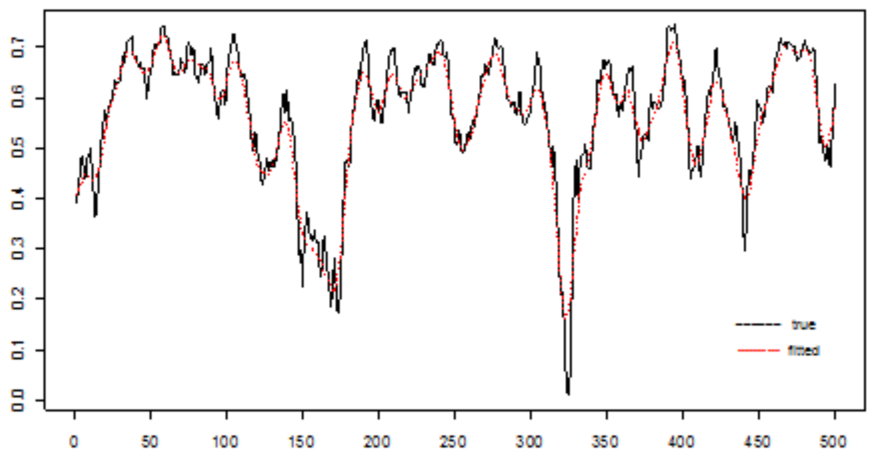


Figure 2. Biases and Mean Square Errors of the Simulation Results

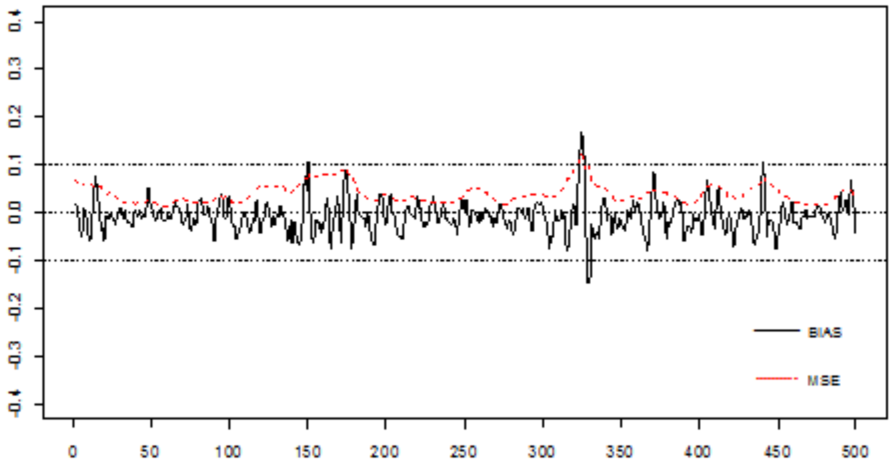
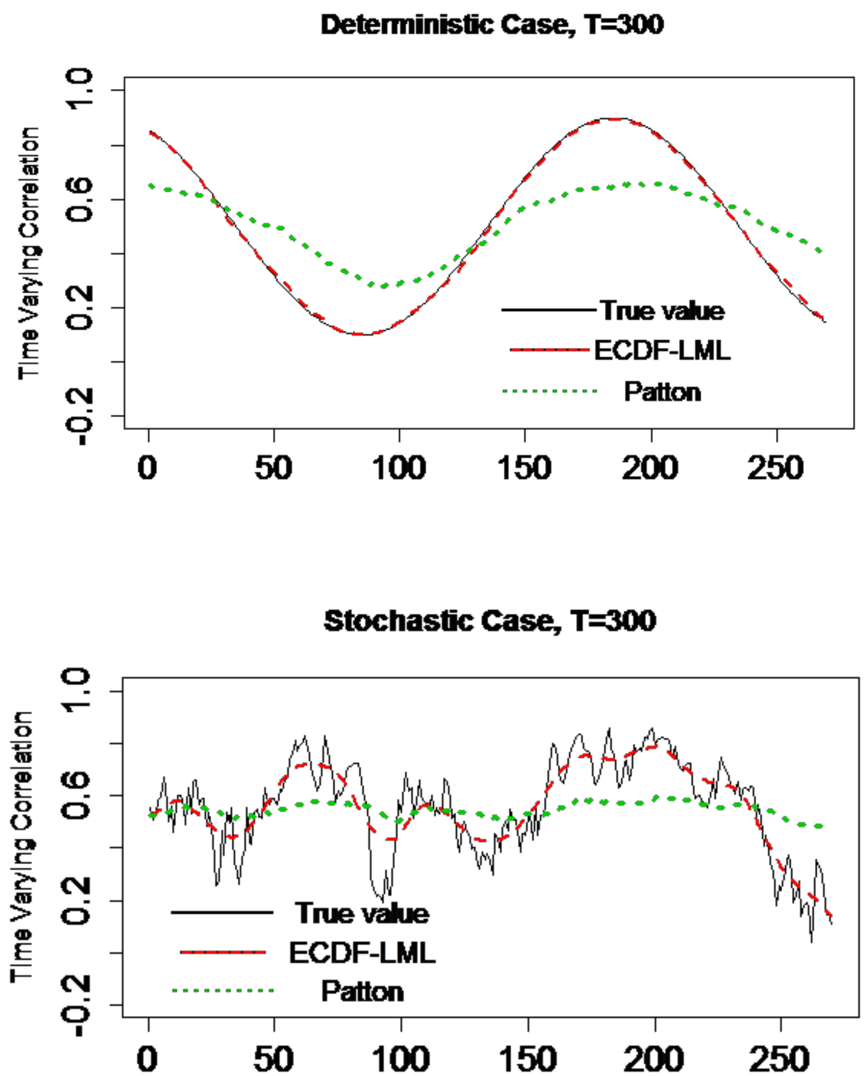


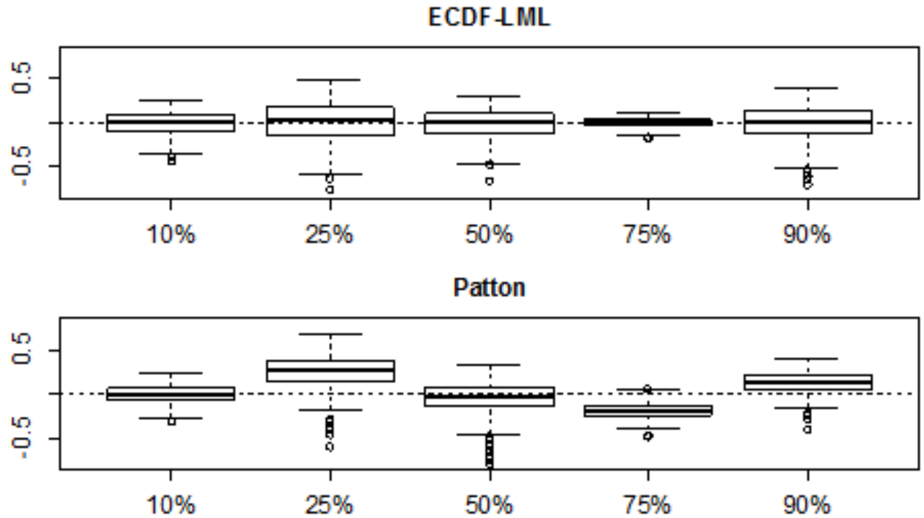
Figure 3. Time-series Plots for the Estimated Copula Parameters



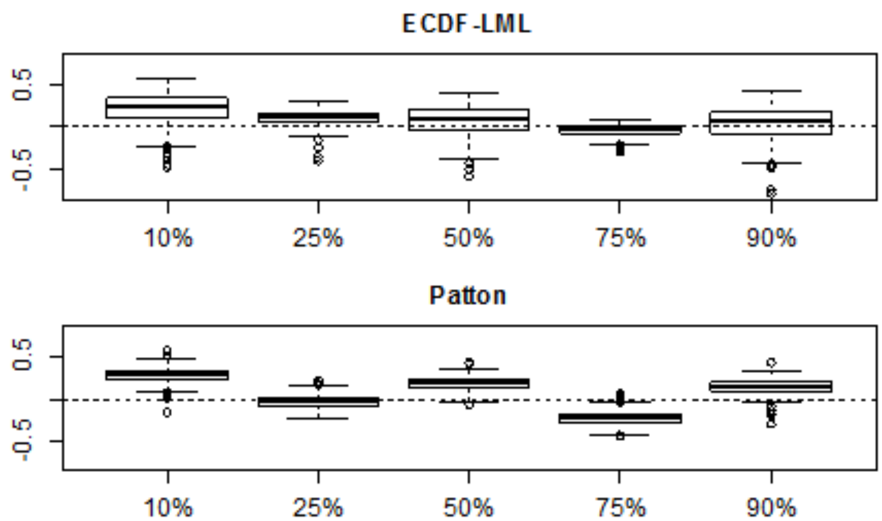
Note: Sample size $T = 300$*, and the number of Monte Carlo replication is 300.*

Figure 4. Boxplots for the Estimated Errors

Deterministic Case, $T = 300$



Stochastic Case, $\mathrm{T} = 300$



Note: Sample size $T = 300$ and the number of Monte Carlo replication is 300.

Figure 5. Shanghai and Shenzhen Time-Varying Correlation

Figure 6. Shanghai and Hong Kong Time-Varying Correlation

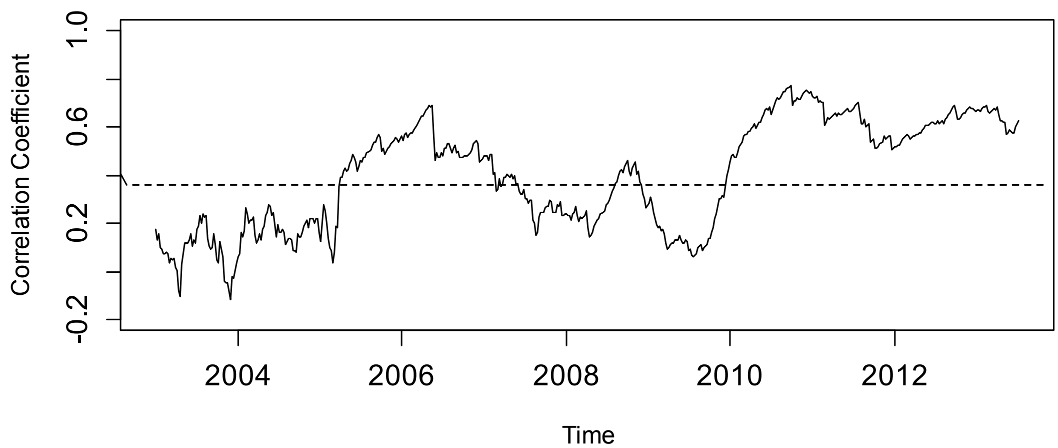


Figure 7. Shenzhen and Hong Kong Time-Varying Correlation