

Some Nonparametric Asymptotic Results for a  
Class of Stochastic Processes

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### **Abstract**

This paper provides a solution of a generalized eigenvalue problem for integrated processes of order 2 in a nonparametric framework. Our analysis focuses on a pair of random matrices related to such integrated process. The matrices are constructed considering some weight functions. Under asymptotic conditions on such weights, convergence results in distribution are obtained and the generalized eigenvalue problem is solved. Differential equations and stochastic calculus theory are used.

**Keywords:** Generalized eigenvalue problem, Differential Equations, Asymptotic theory, Nonparametric analysis.

**Mathematics Subject Classification (2000):** 62E20, 62G20.

# 1 Introduction

Nonparametric approaches have been recently proposed to study integrated processes of order one (Bierens, 1997, Breitung, 2002 and García and Sansó, 2006). The prominent case of system integrated of higher order is the one of systems integrated of order two,  $I(2)$ . The aim of this paper is to provide a nonparametric theoretical analysis of a multivariate integrated process of order two via asymptotic solution of a generalized eigenvalue problem. Many multivariate techniques such as principal component analysis (Cadima and Jolliffe, 1995, Sun, 2000, Schott, 2006, Fujikoshi et al., 2007 and Boente et al., 2008), correspondence analysis (Leeuw, 1982, Van de Velden and Neudecker, 2000) canonical correlation (Nielsen, 2001), discriminant analysis (Bensmail and Celeux, 1996, Demira and Oz Mehmetb, 2005) and factor analysis (Forni et al., 2005) can be formulated as eigenvalue problems, including generalized eigenvalue problems. In this paper the generalized eigenvalue problem involves two random matrices that take into account the stationary and nonstationary properties of a  $p$ -variate integrated process of order 2, i.e.

$$Y_t = \Delta^{-2}\epsilon_t = (1 - L)^{-2}\epsilon_t, \quad (1)$$

where  $p \in \mathbf{N}$ ,  $Y_t = (Y_t^1, \dots, Y_t^p)$ ,  $\epsilon_t = (\epsilon_t^1, \dots, \epsilon_t^p)$  is a zero-mean stationary process,  $L$  is the lag operator, i.e.  $L\epsilon_t := \epsilon_{t-1}$ , and  $\Delta := 1 - L$ . If  $Y_t \sim I(2)$ , then  $Y_t - Y_0 \sim I(2)$ . Without loss of generality, we assume that  $Y_0 = 0$ .

The random matrices are weighted with functions belonging to certain functional spaces. Under some regularity conditions on the weights, we obtain the convergence of the ordered generalized eigenvalues to random numbers independent on the integrated process. Such random quantities are the ordered solution

of a nonparametric generalized eigenvalue problem.

The paper is organized as follows. Section 2 describes the data generating process. In section 3 the weight functions and the random matrices are defined. In section 4, convergence results for the generalized eigenvalue problem are derived. Section 5 concludes.

## 2 Data generating process

If  $Y_t$  in (1) satisfies the hypotheses of the Wold Decomposition Theorem, then there exists a  $p$ -squared matrix of lag polynomials in the lag operator  $L$  such that

$$\epsilon_t = \sum_{j=0}^{\infty} C_j v_{t-j} =: C(L)v_t, \quad t = 1, \dots, n, \quad (2)$$

where  $v_t$  is a  $p$ -variate stationary white noise process.

### Assumption 1

The process  $\epsilon_t$  can be written as in (2), where  $v_t$  are i.i.d. zero-mean  $p$ -variate gaussian variables with variance equals to the identity matrix of order  $p$ ,  $I_p$ , and there exist  $C_1(L)$  and  $C_2(L)$   $p$ -squared matrices of lag polynomials in the lag operator  $L$  such that all the roots of  $\det C_1(L)$  are outside the complex unit circle and  $C(L) = C_1(L)^{-1}C_2(L)$ .

The lag polynomial  $C(L) - C(1)$  attains value zero at  $L = 1$  with algebraic multiplicity equals to 2. Thus, there exists a lag polynomial

$$D(L) = \sum_{k=0}^{\infty} D_k L^k$$

such that  $C(L) - C(1) = (1 - L)^2 D(L)$ . Therefore, we can write

$$\epsilon_t = C(L)v_t = C(1)v_t + [C(L) - C(1)]v_t = C(1)v_t + D(L)(1 - L)^2 v_t. \quad (3)$$

Let us define  $w_t := D(L)v_t$ . Then, substituting  $w_t$  into (3), we get

$$\epsilon_t = C(1)v_t + (1 - L)^2 w_t. \quad (4)$$

(4) implies that, given  $Y_t \sim I(2)$ , we can write recursively

$$\Delta Y_t = \Delta Y_{t-1} + \epsilon_t = \Delta Y_0 + (1 - L)w_t - w_0 + C(1) \sum_{j=1}^t v_j \quad (5)$$

where  $\text{rank}(C(1)) = p - r < p$ .

**Remark 1.** *By Assumption 1, we have that  $C(L)v_t$  and  $D(L)v_t$  are well-defined stationary processes.*

### Assumption 2

Let us consider  $R_r$  the matrix of the eigenvectors of  $C(1)C(1)^T$  corresponding to the  $r$  zero eigenvalues. Then the matrix  $R_r^T D(1)D(1)^T R_r$  is nonsingular.

**Remark 2.** *Assumption 2 implies that  $Y_t$  cannot be integrated of order  $\bar{d}$ , with  $\bar{d} > 2$ . In fact, if there exists  $\bar{d} > 2$  such that  $Y_t \sim I(\bar{d})$ , then the lag polynomial  $D(L)$  admits a unit root with algebraic multiplicity  $\bar{d} - 2$ , and so  $D(1)$  is singular. Therefore  $R_r^T D(1)D(1)^T R_r$  is singular, and Assumption 2 does not hold.*

## 3 Weighted random matrices

In order to address the solution of the generalized eigenvalues problem, a couple of random matrices are constructed. These matrices are associated with the stationary and nonstationary part of the process  $I(2)$ .

If  $Y_t$  satisfies (1), then  $\Delta^k Y_t$  is a nonstationary process, for  $k = 0, 1$  and  $\Delta^2 Y_t$  is a stationary process.

The random matrices are assumed to be dependent on an integer number  $m \geq p$ .

Let us fix  $k = 1, \dots, m$ . We define

$$A_m := \sum_{k=1}^m a_{n,k} a_{n,k}^T; \quad (6)$$

$$B_m := \sum_{k=1}^m b_{n,k} b_{n,k}^T, \quad (7)$$

where

$$a_{n,k} := \frac{M_n^{Y,\Delta Y} / \sqrt{n}}{\sqrt{\int_0^1 \int_0^1 F_k(x) F_k(y) \min\{x, y\} dx dy}}; \quad (8)$$

$$b_{n,k} := \frac{\sqrt{n} M_n^{\Delta^2 Y}}{\sqrt{\int_0^1 F_k(x)^2 dx}}, \quad (9)$$

with

$$M_n^{Y,\Delta Y} = \frac{1}{n} \sum_{t=1}^n \left[ \left( G_k(t/n) + \frac{H_k(t/n)}{n^3} \right) \cdot Y_t \right] + \frac{1}{n} \sum_{t=1}^n F_k(t/n) \Delta Y_t; \quad (10)$$

$$M_n^{\Delta^2 Y} = \frac{1}{n} \sum_{t=1}^n F_k(t/n) \Delta^2 Y_t, \quad (11)$$

where

$$F_k : [0, 1] \rightarrow \mathbf{R}, \quad F_k \in C^1[0, 1];$$

$$G_k : [0, 1] \rightarrow \mathbf{R};$$

$$H_k : [0, 1] \rightarrow \mathbf{R}.$$

The weights  $F_k$ ,  $G_k$  and  $H_k$  can be chosen in order to obtain convergence results for the random matrices  $A_m$  and  $B_m$ . We give the following definition.

**Definition 3.** *Let us fix  $m \in \mathbf{N}$ ,  $k = 1, \dots, m$ . Consider the following conditions:*

$$\lim_{n \rightarrow +\infty} n \cdot \max_{1 \leq t \leq n} \left| \frac{t(t+1)}{2} G_k(t/n) - t F_k(t/n) \right| = 0; \quad (12)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n^{\frac{9}{2}}} \sum_{t=1}^n t(t+1) H_k(t/n) = 0; \quad (13)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^n F_k(t/n) = 0; \quad (14)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n\sqrt{n}} \sum_{t=1}^n t F_k(t/n) = 0; \quad (15)$$

$$\int_0^1 \int_0^1 F_i(x) F_j(y) \min\{x, y\} dx dy = 0, \quad i \neq j; \quad (16)$$

$$\int_0^1 F_i(x) \int_0^x F_j(y) dx dy = 0, \quad i \neq j; \quad (17)$$

$$\int_0^1 F_i(x) F_j(x) dx = 0, \quad i \neq j. \quad (18)$$

The functional classes  $\mathcal{F}_m$ ,  $\mathcal{G}_m$  and  $\mathcal{H}_m$  are

$$\mathcal{F}_m := \left\{ F_k : [0, 1] \rightarrow \mathbf{R}, F_k \in C^1(0, 1) \mid (14) - (18) \text{ hold, } k = 1 \dots, m \right\}; \quad (19)$$

$$\mathcal{G}_m := \left\{ G_k : [0, 1] \rightarrow \mathbf{R} \mid (12) \text{ holds, } k = 1 \dots, m \right\}; \quad (20)$$

$$\mathcal{H}_m := \left\{ H_k : [0, 1] \rightarrow \mathbf{R} \mid (13) \text{ holds, } k = 1 \dots, m \right\}. \quad (21)$$

(Bierens, 1997) shows that the functional class  $\mathcal{F}_m$  is not empty. He pointed out that, if one defines

$$\bar{F}_k : \mathbf{R} \rightarrow \mathbf{R}$$

such that

$$\bar{F}_k(x) = \cos(2k\pi x), \quad (22)$$

and taking the restriction

$$F_k := \bar{F}_k|_{[0,1]},$$

then  $F_k \in \mathcal{F}_m$ .

The functional classes  $\mathcal{G}_m$  and  $\mathcal{H}_m$  are also not empty. In fact, the following result holds.

**Proposition 4.** Fix  $k = 1, \dots, m$ . Define the following subset of  $\mathbf{R}$ :

$$A := \bigcup_{n \in \mathbf{N}} \left\{ x \in \mathbf{R} \mid x = -\frac{1}{n} \right\},$$



and the functions

$$\bar{G}_k : \mathbf{R} - A \rightarrow \mathbf{R},$$

$$\gamma : \mathbf{N} \rightarrow \mathbf{R},$$

such that

$$\bar{G}_k(x) = \frac{k\pi x + 1}{nx + 1} + \gamma(n). \quad (23)$$

Moreover, define

$$\bar{H}_k : \mathbf{R} \rightarrow \mathbf{R}$$

such that

$$\bar{H}_k(x) = \sum_{j=1}^N a_j x^{\alpha_j}, \quad (24)$$

for each  $N \in \mathbf{N}$ ,  $a_j, \alpha_j \in \mathbf{R}$ ,  $\forall j \in \{1, \dots, N\}$ .

Assume that:

- the function

$$f : \mathbf{R} - \{-1\} \rightarrow \mathbf{R}$$

such that

$$f(t) := \frac{t(t+1)}{2} \bar{G}_k(t/n) - t \cos\left(\frac{2k\pi t}{n}\right) \quad (25)$$

is increasing with respect to  $t$ ;

- the function  $\gamma$  satisfies the following condition:

$$n \cdot \max \left\{ \left| \frac{1}{2} \bar{G}_k(1/n) - \cos\left(\frac{2k\pi}{n}\right) \right|, \left| \frac{n(n+1)}{2} \bar{G}_k(1) - n \right| \right\} = o\left(\frac{1}{n}\right). \quad (26)$$

Then

$$G_k := \bar{G}_k|_{[0,1]}, \quad H_k := \bar{H}_k|_{[0,1]}$$

belong to  $\mathcal{G}_m$  and  $\mathcal{H}_m$ , respectively.

*Proof.* A direct computation gives that  $H_k \in \mathcal{H}_m$ . So we have to prove that  $G_k \in \mathcal{G}_m$ .

Since  $F_k$  defined in (22) belongs to  $\mathcal{F}_m$  (see Bierens, 1997), we can replace in (12) the functions  $F_k$  with (22). We get

$$\lim_{n \rightarrow +\infty} n \cdot \max_{1 \leq t \leq n} \left| \frac{t(t+1)}{2} G_k(t/n) - t \cos\left(\frac{2k\pi t}{n}\right) \right| = 0. \quad (27)$$

Then there exists  $\epsilon > 0$  such that

$$\max_{1 \leq t \leq n} \left| \frac{t(t+1)}{2} G_k(t/n) - t \cos\left(\frac{2k\pi t}{n}\right) \right| \sim \frac{1}{n^{1+\epsilon}}. \quad (28)$$

Let us consider  $f$  defined as in (25). Since  $f$  is increasing, a simple estimate gives

$$\begin{aligned} f'(t) &:= \frac{2t+1}{2} G_k(t/n) + \frac{t(t+1)}{2n} \frac{\partial}{\partial t} G_k(t/n) - \cos\left(\frac{2k\pi t}{n}\right) + \frac{2k\pi t}{n} \sin\left(\frac{2k\pi t}{n}\right) > \\ &> \frac{2t+1}{2} G_k(t/n) + \frac{t(t+1)}{2n} \frac{\partial}{\partial t} G_k(t/n) - 1 - \frac{2k\pi t}{n} \geq 0. \end{aligned}$$

Thus, the weight functions  $G_k$  can be obtained by solving the differential equation

$$\frac{2t+1}{2} G_k(t/n) + \frac{t(t+1)}{2n} \frac{\partial}{\partial t} G_k(t/n) - 1 - \frac{2k\pi t}{n} = 0. \quad (29)$$

The solution of (29) is

$$G_k(t/n) = \frac{1}{n} \cdot \frac{k\pi(t/n) + 1}{t/n + 1/n} + \gamma(n),$$

where  $\gamma$  is independent on  $t$ . Due to the fact that  $f$  is increasing with respect to  $t$ , the condition (26) implies that (12) holds.  $\square$

## 4 Generalized eigenvalues and nonparametric results

In this section the generalized eigenvalue problem is solved. Consider a  $p$ -variate standard Wiener process  $W$  and denote with  $f_k$  the derivative of  $F_k$ . We define the following  $p$ -variate standard normally distributed random vectors:

$$\begin{aligned} X_k &:= \frac{\int_0^1 F_k(x)W(x)dx}{\left(\int_0^1 \int_0^1 F_k(x)F_k(y) \min\{x, y\}dxdy\right)^{\frac{1}{2}}}, \\ Y_k &:= \frac{F_k(1)W(1) - \int_0^1 f_k(x)W(x)dx}{\int_0^1 F_k(x)^2 dx}, \\ X_k^* &:= \left(R_{p-r}^T C(1)C(1)^T R_{p-r}\right)^{\frac{1}{2}} R_{p-r}^T C(1)X_k \sim N_{p-r}(0, I_{p-r}), \\ Y_k^* &:= \left(R_{p-r}^T C(1)C(1)^T R_{p-r}\right)^{\frac{1}{2}} R_{p-r}^T C(1)Y_k, \\ Y_k^{**} &:= (R_r^T D(1)D(1)^T R_r)^{-\frac{1}{2}} R_r^T D(1)Y_k \sim N_r(0, I_r). \end{aligned}$$

Furthermore, we construct the matrix  $V_{r,m}$  as

$$V_{r,m} := (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}} V_{r,m}^* (R_r^T D(1)D(1)^T R_r)^{\frac{1}{2}},$$

where

$$V_{r,m}^* = \left(\sum_{k=1}^m \gamma_k^2 Y_k^{**} Y_k^{**T}\right) - \left(\sum_{k=1}^m \gamma_k Y_k^{**} X_k^{*T}\right) \left(\sum_{k=1}^m X_k^* X_k^{*T}\right)^{-1} \left(\sum_{k=1}^m \gamma_k X_k^* Y_k^{**T}\right).$$

**Theorem 5.** *Assume that  $F_k \in \mathcal{F}_m$ ,  $G_k \in \mathcal{G}_m$ ,  $H_k \in \mathcal{H}_m$  and Assumptions 1 and 2 hold.*

- Let us consider  $\hat{\lambda}_{1,m} \geq \dots \geq \hat{\lambda}_{p,m}$  the ordered solutions of the generalized eigenvalue problem

$$\det\left[A_m - \lambda(B_m + n^{-2}A_m^{-1})\right] = 0, \quad (30)$$

and let us consider  $\lambda_{1,m} \geq \dots \geq \lambda_{p-r,m}$  the ordered solutions of the generalized eigenvalue problem

$$\det \left[ \sum_{k=1}^M X_k^* X_k^{*T} - \lambda \sum_{k=1}^M Y_k^* Y_k^{*T} \right] = 0, \quad (31)$$

where the  $X_i^*$ 's and  $Y_j^*$ 's are i.i.d. random variables following a  $N_{p-r}(0, I_{p-r})$  distribution.

Then we have the following convergence in distribution

$$(\hat{\lambda}_{1,m}, \dots, \hat{\lambda}_{p,m}) \rightarrow (\lambda_{1,m}, \dots, \lambda_{p-r,m}, 0, \dots, 0).$$

- Let us consider  $\lambda_{1,m}^* \geq \dots \geq \lambda_{r,m}^*$  the ordered solutions of the generalized eigenvalue problem

$$\det \left[ V_{r,m}^* - \lambda (R_r^T D(1) D(1)^T R_r)^{-1} \right] = 0. \quad (32)$$

We have the following convergence in distribution

$$n^2 (\hat{\lambda}_{p-r+1,m}, \dots, \hat{\lambda}_{p,m}) \rightarrow (\lambda_{1,m}^{*2}, \dots, \lambda_{r,m}^{*2}).$$

*Proof.* Due to (Anderson et al., 1983), then Lemmas 1, 2 and 4 in (Bierens, 1997), it is sufficient to prove that

$$\frac{M_n^{Y, \Delta Y}}{\sqrt{n}} \rightarrow C(1) \int_0^1 F_k(x) W(x) dx, \quad \text{as } n \rightarrow +\infty. \quad (33)$$

By definition of data generating process, we can write

$$M_n^{Y, \Delta Y} = \frac{1}{n} \sum_{t=1}^n \left( G_k(t/n) + \frac{H_k(t/n)}{n^3} \right) \left[ \sum_{j=0}^{t-1} \Delta Y_{t-j} \right] + \frac{1}{n} \sum_{t=1}^n F_k(t/n) \Delta Y_t. \quad (34)$$

We get recursively

$$\sum_{j=0}^{t-1} \Delta Y_{t-j} = \sum_{j=0}^{t-1} (j+1) \epsilon_{t-j} \sim N(0, \Sigma_*), \quad (35)$$

where

$$\Sigma_* := \begin{pmatrix} \sum_{j=0}^{t-1} (j+1)^2 \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sum_{j=0}^{t-1} (j+1)^2 \sigma_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \sum_{j=0}^{t-1} (j+1)^2 \sigma_p^2 \end{pmatrix}$$

By (35) and (36) we can write

$$M_n^{Y, \Delta Y} = \frac{1}{n} \sum_{t=1}^n \left( G_k(t/n) + \frac{H_k(t/n)}{n^3} \right) \left[ \sum_{j=0}^{t-1} (j+1) \epsilon_{t-j} \right] + \frac{1}{n} \sum_{t=1}^n F_k(t/n) \Delta Y_t. \quad (36)$$

Thus, (36) can be rewritten. Using the definition of the  $p$ -variate normal random variable  $\epsilon_t$  and the i.i.d. property, we get

$$\begin{aligned} \frac{M_n^{Y, \Delta Y}}{\sqrt{n}} &= \frac{\epsilon_1}{n^4 \sqrt{n}} \sum_{t=1}^n H_k(t/n) \frac{t(t+1)}{2} + \\ &+ \frac{\epsilon_1}{n \sqrt{n}} \cdot \left[ \sum_{t=1}^n \left( G_k(t/n) \frac{t(t+1)}{2} + t F_k(t/n) \right) \right] \end{aligned} \quad (37)$$

By hypothesis (13), the first addend in the right-hand term of (37) vanishes as  $n \rightarrow +\infty$ .

Moreover, since  $G_k \in \mathcal{G}_m$  it results, for each  $t = 1, \dots, n$ ,

$$G_k(t/n) \frac{t(t+1)}{2} \sim t F_k(t/n), \quad (38)$$

as  $n \rightarrow +\infty$ .

Therefore, since  $F_k \in \mathcal{F}_m$ , by (38) and theorems 1 and 2 in (Bierens, 1997), we get the thesis.  $\square$

## 5 Conclusions

This paper provides a nonparametric analysis of multivariate integrated processes of order two via the asymptotic behavior of a generalized eigenvalue

problem. Two involved random matrices associated with the stationary and nonstationary parts of the process are constructed. To obtain asymptotic results, some weights regarding the matrices are considered. The ordered generalized eigenvalues converge to some random numbers. Such random quantities are the ordered solution of a generalized eigenvalue problem independent on the data generating process.

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