Three Problems in Robotics

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Abstract

Three rather different problems in robotics are studied using the same technique from screw theory. The first problem concerns systems of springs. The potential function is differentiated in the direction of an arbitrary screw to find the equilibrium position. The second problem is almost identical in terms of the computations, the least squares solution to the problem of finding the rigid motion undergone by a body given only data about points on the body is sought. In the third problem the Jacobian of a Stewart platform is found. Again this is achieved by differentiating with respect to a screw. Further, second order properties of the first two problems are studied. The Hessian of second derivatives is computed and hence the stability properties of the equilibrium positions of the spring system are found.

Notation

$\mathbb{R}^n$ n-dimensional Euclidean space.

$\mathbb{RP}^3$ 3-dimensional projective space.

$SE(3)$ The group of proper Euclidean motions in 3 dimensions, that is rigid body motions.

$SO(3)$ The group of rotations in 3 dimensions.

R A 3×3 rotation matrix.

M A 4×4 matrix representing a rigid transformation, sometimes called the homogeneous representation.

$\mathbf{a}, \mathbf{b}$ The position vectors of points in 3-dimensional space.

$\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$ Points in 3-dimensional space represented as points in 4-dimensional space, in partitioned form, $\tilde{\mathbf{a}}^T = (\mathbf{a}^T, 1)$ for example.

A, B 3-dimensional vectors represented as 3×3 anti-symmetric matrices, so that $A\mathbf{x} = \mathbf{a} \times \mathbf{x}$ for any vector $\mathbf{x}$.

I_3 The 3×3 identity matrix.

λ Spring constant.

J 6×6 Jacobian matrix.

K 6×6 stiffness matrix.

S 4×4 matrix representing a Lie algebra element or screw.

$\mathcal{W}$ 6×1 vector representing a wrench, that is an element of the vector space dual to the Lie algebra.

1 Introduction

When Ball wrote his treatise [1] at the end of the 19th centaury Sophius Lie was writing about what he called ‘continuous groups’. If Ball knew of Lie’s work it may not have been obvious that it had any connection to his own since Lie was interested in symmetries of differential equations. It was Klein who later introduced the idea that these ‘Lie groups’ could be thought of as geometrical symmetries. It was after both Ball and Lie had died that ‘Lie theory’ began to finds it place as central to modern geometry. In particular, the work of Killing and Cartan on Lie algebras were very influential. For more details of the history of Lie theory see [2]. With hindsight it can be seen that Ball’s finite screws were simply elements of a Lie group: the group of proper Euclidean transformations in $\mathbb{R}^3$. The twists or motors were elements of the Lie algebra of this group. These are also sometimes referred to as infinitesimal screws although more precisely Ball’s screws can be identified with elements of the projective space formed from the Lie algebra. That is the lines through the origin in the Lie algebra.

Many other elements of Lie theory were also present in Ball’s screw theory. But perhaps their significance was not fully appreciated. For example, the Lie product or Lie bracket is simply the cross product of screws. For Ball this was just a geometrical operation, the analogue of the vector product of 3-dimensional vectors.

Some authors refer to Screw theory and Lie group methods as if they were different approaches. The view of this author is that there is no distinction between them, screw theory is simply the specialisation of Lie theory to the group of rigid body transformations. However, the name screw theory remains useful. As a descriptive shorthand and also it is a reminder that it was Ball who worked out almost all of the theory before Lie groups were invented!

A great deal of robotics is concerned with rigid motions. In kinematics and dynamics the rigid motions both the payload and the links of the robot are studied. In robot vision a common problem is to retrieve the rigid motion experienced by the camera from the images it has taken. Hence, the group of rigid body motions is a central object in robotics. To date screw theory has been much used in robot kinematics where it was introduced by the mechanisms community. It is beginning to be used in the dynamics and control of robots but is by no means the method of first choice in these areas. In robot vision these techniques are hardly known. One of the aims of this work is to demonstrate that these methods have a universal applicability to problems in robotics and to show that a wide variety of problems in robotics share an underlying theme.

This work uses the fact that a Lie group is a differential manifold. To minimise a smooth function on such a space the machinery of Lagrange multipliers is not needed. It is possible to work on the manifold directly, it is not necessary to think of the group as embedded in Cartesian space, as would be implied by the use of Lagrange multipliers.

Differentiation along tangent vector fields can be used to find equations for a the stationary points of a function. The most convenient vector fields to use are the left-invariant fields on the group. These are simply elements of the Lie algebra of the group,

Figure 1: A Rigid Body Suspended by a System of Springs

the twists or screws. Hence this technique could be thought of as, “differentiating along a screw”.

2 Springs

Consider a rigid body supported by a system of springs, see figure 1. Further assume that the springs have natural length 0, obey Hooke’s law and can both push and pull. The spring constants λ_i, of the springs can be different. Gravity will be ignored here for simplicity. However it is not difficult to see how it could be incorporated in a more sophisticated model, either by modifying the potential function that is derived below or by changing the equations for equilibrium to include the wrench due to gravity.

Let $\tilde{\mathbf{a}}_i^T = (\mathbf{a}_i^T, 1)$ be the points where the springs are attached to the ground or frame, and $\tilde{\mathbf{b}}_i^T = (\mathbf{b}_i^T, 1)$ the corresponding attachment points on the rigid body when the body is in some standard ‘home’ configuration. If the body undergoes a rigid motion the attachment points will move to,

$$\begin{pmatrix} \mathbf{b}_i' \\ 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{b}_i \\ 1 \end{pmatrix},$$

which will be abbreviated to $\tilde{\mathbf{b}}_i' = M\tilde{\mathbf{b}}_i$. The first question that can be asked about this situation is: Is there an equilibrium configuration for the rigid body?

The problem is to minimise the potential energy of the spring system, this is given by the following function,

$$\Phi = \frac{1}{2}\sum_i \lambda_i \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right)^T \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right).$$

Notice that this function is defined on the group $SE(3)$, that is $\Phi : SE(3) \longrightarrow \mathbb{R}$, as M varies over the group different values of the potential result.

If this was just a function on $\mathbb{R}^n$ the stationary points would be found by calculating the partial derivatives and then setting them to zero. The standard method of tackling this problem would be to minimise the matrix elements using Lagrange multipliers to take account of the constraint that the matrix must be a group element.

However, the simpler method for unconstrained functions can be imitated using some manifold theory. To find the stationary points of a function defined on a manifold the function must be differentiated along vector fields on the manifold. As usual the results is set to zero and then the resulting equations solved to find the stationary points. For this to work a set of vector fields which span the space of all vector field on the manifold is required.

As the manifold under consideration is the underlying manifold of a Lie group, such a complete set of vector fields is always available. The elements of the Lie algebra thought of as left invariant vector fields can be used.

To differentiate along a vector field the value of the function at two points is compared, the current position and a position a little distance along a path tangent to the vector field, then the limit of the difference between values of the function at these neighbouring points is taken as the path gets shorter and shorter.

Let:

$$S = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix},$$

be a Lie algebra element or screw given in the 4×4 representation. Here Ω is a 3×3 anti-symmetric matrix corresponding to a vector $\boldsymbol{\omega}$, that is $\Omega \mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$ for any vector $\mathbf{x}$.

If M is a group element written in the 4×4 representation, then the action of S on M, is given by the left translation,

$$M(t) = e^{tS} M.$$

This takes M along a path tangent to the vector field defined by S. Taking the derivative along the path and then setting $t = 0$ gives,

$$\partial_S M = SM.$$

Hence the derivative of the potential is given by,

$$\partial_S \Phi = -\sum_i \lambda_i \left(\tilde{\mathbf{a}}_i - M \tilde{\mathbf{b}}_i \right)^T SM \tilde{\mathbf{b}}_i.$$

For equilibrium this must vanish for arbitrary S. Hence S must be separated out, so consider the term,

$$SM\tilde{\mathbf{b}}_i = \begin{pmatrix} \boldsymbol{\omega} \times (R\mathbf{b}_i + \mathbf{t}) + \mathbf{v} \\ 0 \end{pmatrix} = \begin{pmatrix} -RB_iR^T - T & I_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}.$$

where I_3 is the 3×3 identity matrix and B_i is the anti-symmetric matrix corresponding to $\mathbf{b}_i$.

Substituting this into the equilibrium equation and using the fact that S and thus $\boldsymbol{\omega}$ and $\mathbf{v}$ are arbitrary, the following result is obtained,

$$\sum_i \lambda_i \left(\tilde{\mathbf{a}}_i - M \tilde{\mathbf{b}}_i \right)^T \begin{pmatrix} -RB_iR^T - T & I_3 \\ 0 & 0 \end{pmatrix} = 0.$$

After a little manipulation, this matrix equation produces 2 vector equations,

$$\sum_i \lambda_i \mathbf{a}_i \times (\mathbf{a}_i - R\mathbf{b}_i - \mathbf{t}) = 0 \tag{1}$$

and

$$\sum_i \lambda_i (\mathbf{a}_i - R\mathbf{b}_i - \mathbf{t}) = 0 \tag{2}$$

If the weights λ_i are all equal, then equation 2 says that the optimal transformation maps the centroids of the *a* points to the *b* points. Another way of putting this is that at an equilibrium configuration the centroids of the two sets of points must coincide. To proceed, choose the origin of coordinates so that the centroid of the *b* points lies at the origin, $\sum_i \lambda_i \mathbf{b}_i = \mathbf{0}$. The translation vector is now given by equation (2) as,

$$\mathbf{t} = \frac{\sum_i \lambda_i \mathbf{a}_i}{\sum_i \lambda_i}.$$

In the above form equation (1) is not very easy to deal with, a more tractable form is the 3×3 representation. A small computation confirms that the anti-symmetric matrix corresponding to a vector product $\mathbf{p} \times \mathbf{q}$, is given by $\mathbf{q}\mathbf{p}^T - \mathbf{p}\mathbf{q}^T$. Hence in this form the equation becomes,

$$\sum_i \lambda_i \left(R\mathbf{b}_i \mathbf{a}_i^T - \mathbf{a}_i \mathbf{b}_i^T R^T \right) = 0,$$

where the result that $\mathbf{t} = \sum_i \lambda_i \mathbf{a}_i / \sum_i \lambda_i$ has been used. Now, writing, $P = \sum_i \lambda_i \mathbf{a}_i \mathbf{b}_i^T$, this equation becomes;

$$RP^T = PR^T \tag{3}$$

This shows that the matrix PR^T is symmetric. So, let $PR^T = Q$ where Q is symmetric, then,

$$P = QR.$$

This decomposes the matrix P as the product of a symmetric matrix with a proper orthogonal one. This is essentially the polar decomposition of the matrix. Notice that the polar decomposition $P = RQ'$ also satisfies the equation, the rotation matrix R here is the same as above but the symmetric matrix $Q' = R^T QR$ is simply congruent to the original symmetric matrix. So as far as the solution for R is concerned there is no difference between these solutions. In fact the polar decomposition of a matrix splits it into an orthogonal matrix and a non-negative symmetric matrix. Here a proper orthogonal matrix and a symmetric one are required. If the orthogonal matrix from the polar decomposition of P is a reflection then multiplying by -1 gives a rotation. More details on the polar decomposition of a matrix can be found in [3], for example.

The polar decomposition gives one solution, but this solution is not unique. Let $P = QR_p$ be the polar decomposition of P, now substitute this into the equation (3),

$$RR_p^T Q = QR_p R^T.$$

Writing $R_i = RR_p^T$ the equation becomes,

$$R_i Q R_i = Q.$$

Suppose that $\mathbf{v}$ lies in the direction of the rotation axis of R_i, so that $R_i\mathbf{v} = \mathbf{v}$, postmultiplying the above equation by $\mathbf{v}$ gives,

$$R_i Q\mathbf{v} = Q\mathbf{v}.$$

Hence $Q\mathbf{v}$ lies along the axis of R_i and so,

$$Q\mathbf{v} = \mu\mathbf{v},$$

for some constant μ. Any solution for the rotation R_i must have its axis of rotation aligned with an eigenvector of Q.

The possible angles of rotation can be found by considering the action on the eigenvectors of Q, using the fact that the eigenvectors of a symmetric matrix are mutually orthogonal. If the matrix P is non-singular then it is well known that the polar decomposition is unique. If the eigenvalues of Q are all different and also have different magnitudes then the only possible angles are 0 and π. This gives four solutions in all, $R_i = I_3$ is the solution found above, that is R is simply the rotation from the polar decomposition. The three other solutions for R_i are rotations of π radians about the three eigenvectors of Q. So in all there are four solutions for the rotation $R = R_p R_i$. In each case R_p is the rotation from the polar decomposition of P and R_i are as above, rotations of π about the ith eigenvector of Q, with the fourth solution given by $R_0 = I_3$. Notice that the four rotations form a discrete subgroup of the group of rotations, this subgroup is the well known Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

If any of the above conditions is broken, P is singular, two of the eigenvalues of Q are equal, or a pair of eigenvalues sum to zero, then there are more solutions. For example, if a pair of eigenvalues of Q sum to zero then any rotation about the remaining eigenvector will satisfy the equation for R_i.

The fact that, in the general case, four stationary points of the potential energy function have been found is not surprising. Morse theory studies the relationship between manifolds and the critical points of functions defined on them, see [4]. The critical points, or stationary points here, correspond to cells in a cellular decomposition of the manifold. The manifold in question here is the underlying manifold of the rotation group $SO(3)$, this is known to be 3-dimensional projective space, $\mathbb{RP}^3$. The minimal cellular decomposition of $\mathbb{RP}^3$ has four cells, with dimensions $0,1,2$ and 3 see [5, p. 105]. Hence, a general function on $SO(3)$ will have a minimum of 4 critical points. Moreover, the index of the critical points, the number of negative eigenvalues of its Hessian matrix, gives the dimension of the corresponding cell. Thus, without any further computations, the four critical points will be known to comprise a local maximum, a local minimum and two types of saddle points. The problem of finding which solution is the minimum will be addressed later.

The above shows that in general, that is when the matrix P is non-singular, the spring system of figure 1 has a unique stable equilibrium position. Moreover, the system will have three unstable equilibrium positions. This result does not depend on the number or arrangement of the springs as long as $\det(P) \neq 0$.

3 Rigid motion from point data

Consider a vision system or range-finding system, which can measure the location of points in 3-dimensions. Imagine that a rigid body has a number of points with known position. The body is subjected to an unknown rigid motion and the positions of the points are measured. These measurements will contain errors and the question to be addressed here is: How can the rigid motion undergone by the body be estimated?

Let the positions of the known points be $\tilde{\mathbf{b}}_i$ and corresponding measured points $\tilde{\mathbf{a}}_i$. Write the unknown rigid transformation as M, then the function,

$$\Phi = \sum_i \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right)^T \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right),$$

represents the sum of squares of the differences between the measured points and their ideal (noise-free) positions. Choosing M to minimise this function gives a 'least-squares' estimate for the rigid transformation. This function is almost identical to the potential energy function studied in the previous section, the only differences are an overall factor of one half and that all the λ_is have been set to 1.

The history of this problem is very interesting. The problem of finding the rotation is clearly the interesting part and was first solved by MacKenzie in 1957 [6]. He came upon this problem in the context of crystallography. In 1966 Wahba found the same problem while studying the orientation of artificial satellites, [7]. In 1976 Moran re-solved the problem using quaternions, [8]. The motivation here came from geology, in particular the movement of tectonic plates. In the context of manufacturing, Nádas found and re-solved the problem in 1978, [9]. Here the application was to the manufacture of ceramic substrates for silicon chips. In the robot vision community the problem is usually credited to Horn [10], for example see [11, Chap. 5].

The solution given above is, perhaps, a little simpler than the standard arguments which involve a constrained minimisation, the constraints being used to express the fact that the matrices must lie in the group.

The standard solutions have not always been in terms of the polar decomposition. In fact (at least), two other descriptions of the solution are possible.

To compute the polar decomposition of a matrix, texts on numerical analysis recommend beginning with the singular value decomposition of the matrix, see for example [12]. Hence, it is no real surprise that the solution to our problem can be obtained directly from a singular value decomposition.

Recall, from the section above that $RP^T = PR^T$, must be solved for the rotation matrix R, where $P = \sum_i \lambda_i \mathbf{a}_i \mathbf{b}_i^T$. Further, recall that,

$$P = QR_p,$$

where Q was symmetric. Hence, Q can be diagonalised as $Q = UDU^T$ with U orthogonal and D diagonal. Now write,

$$P = UDU^T R_p = UDV^T,$$

where $V^T = U^T R$ is still orthogonal. This is simply the singular value decomposition of P. To put this another way, suppose the singular value decomposition of P is $P = UDV^T$ then the four solutions are $R = UV^T R_i$.

Another form for the solution can be derived as follows, begin with the polar decomposition of P,

$$P = QR_p,$$

where Q is a symmetric matrix. Postmultiplying this equation by its transpose gives, $PP^T = Q^2$. Finally substituting for Q yields,

$$R = (PP^T)^{-1/2} PR_i.$$

There are several different square roots of the matrix (PP^T) that could be taken here, the choice is limited by the requirement that the determinant of R must be 1. This means that the unique positive square root must be taken, see [3, p. 405].

Finally here consider the determinant of the matrix P. A classical result tells us that the polar decomposition of a matrix P is unique if P is non-singular, see [3, p. 413] for example. The classical polar decomposition, decomposes the matrix P into an orthogonal matrix and positive-semidefinite symmetric matrix. Here a proper-orthogonal matrix and a symmetric matrix (not necessarily positive-semidefinite) are needed, this does not effect the uniqueness of the solution.

Thus the determinant of the matrix P defined above, needs to be investigated. To simplify the discussion assume that the spring stiffnesses λ_i have all been set to 1.

When there are less than three springs or pairs of points the determinant is always singular. For three point-pairs a straightforward computation reveals,

$$\det(P) = \det\left(\sum_{i=1}^{3} \mathbf{a}_i \mathbf{b}_i^T\right) = (\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3),$$

here the scalar triple product has been written as, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{abc})$.

Generalising this to n point-pairs gives,

$$\det(P) = \sum_{1\le i<j<k\le n} (\mathbf{a}_i\mathbf{a}_j\mathbf{a}_k)(\mathbf{b}_i\mathbf{b}_j\mathbf{b}_k).$$

Certainly this is singular if all the a points or all the b points lie on a plane through the origin.

4 Jacobian matrix for Stewart platforms

Consider a general Stewart platform. This manipulator has six legs connected in parallel. Each leg consists of an hydraulic actuator between a pair of passive spherical joints. The six legs connect the base or ground to a movable platform. By adjusting the lengths of the six legs using the hydraulic actuators the platform can be manoeuvred with six degrees-of-freedom. See figure 2.

Figure 2: A General Stewart Platform

For parallel manipulators it is the inverse kinematics that is straightforward, while the forward kinematics are difficult. Suppose that the position and orientation of the platform is given, the leg-lengths are simple to find. Let $\mathbf{a}_i$ be the position of the centre of the spherical joint on the ground belonging to the i-th leg. In the home configuration the corresponding position of the joint centre on the platform will be, $\mathbf{b}_i$. So now the length of the i-th leg, or rather its square, can be written,

$$l_i^2 = \big(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\big)^T \big(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\big) \qquad i = 1, \ldots, 6.$$

As usual M is a rigid transformation, this time the motion that takes the platform from home to the current position. Notice that the leg-lengths can be thought of as functions on the group, however it is more usual to think of these as components of a mapping from the group to the space of leg-lengths, $SE(3) \longrightarrow \mathbb{R}^6$. A point in $\mathbb{R}^6$ is given in coordinates as $(l_1, l_2, \ldots, l_6)$. It is the jacobian of this mapping that is sought. To do this the derivative of the leg-lengths is taken,

$$\left.\frac{dl_i^2}{dt}\right|_{t=0} = 2l_i\dot{l}_i = -2\big(\tilde{\mathbf{a}}_i - \tilde{\mathbf{b}}_i\big)^T S\tilde{\mathbf{b}}_i.$$

Rearranging this gives,

$$\dot{l}_i = \frac{1}{l_i}\big(\tilde{\mathbf{b}}_i - \tilde{\mathbf{a}}_i\big)^T S\tilde{\mathbf{b}}_i = \frac{1}{l_i}\big((\mathbf{a}_i \times \mathbf{b}_i)^T, (\mathbf{b}_i - \mathbf{a}_i)^T\big) \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}.$$

This gives the joint rate of each leg as a linear function of the velocity screw of the platform. The jacobian J is the matrix satisfying the formula,

$$\begin{pmatrix} \dot{l}_1 \\ \vdots \\ \dot{l}_6 \end{pmatrix} = J \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}.$$

So it can be seen that the rows of this Jacobian matrix are simply,

$$\frac{1}{l_i}\big((\mathbf{a}_i \times \mathbf{b}_i)^T, (\mathbf{b}_i - \mathbf{a}_i)^T\big), \qquad i = 1, \ldots, 6.$$

This is the wrench given by a unit force directed along the i-th leg.

Consider a system of springs as in section 2. Suppose there are just six springs. Now it can be shown that the Jacobian associated with the equilibrium position is singular. To see this consider equations (1) and (2), which define the equilibrium position, arranging things so that the equilibrium position is the reference position and hence $R = I_3$ and $\mathbf{t} = \mathbf{0}$ then the equations become,

$$\sum_i \lambda_i \mathbf{a}_i \times \mathbf{b}_i = 0,$$

and

$$\sum_i \lambda_i (\mathbf{a}_i - \mathbf{b}_i) = 0.$$

In term of the Jacobian for a corresponding Stewart platform, that is, one whose leg-lengths correspond to the lengths of the springs, it can be seen that the rows of the Jacobian are linearly dependent and hence the matrix is singular.

The forward kinematics problem for a Stewart platform is to determine the position and orientation of the platform given the leg-lengths. It is well known that, for a given set of leg-lengths there are a finite number of different solutions, in general 40. Different solutions are referred to as different poses or postures of the platform. Replacing the legs with springs it is clear that the potential function will have the same value in each of these poses or postures since the function only depends on the lengths of the springs. However, none of these positions will be minima of the potential function since, as has been shown above, the stable equilibrium position is unique.

5 The Stiffness Matrix

The problems presented in the last three sections are well known and have been solved by many different methods. The advantage of the screw theory methods studied here is that it is relatively easy to study higher derivatives.

First note that it is possible to find the wrench due to the springs. In general a wrench is a 6-dimensional vector of forces and torques,

$$\mathcal{W} = \begin{pmatrix} \boldsymbol{\tau} \\ \mathbf{F} \end{pmatrix},$$

where $\boldsymbol{\tau}$ is a moment about the origin and $\mathbf{F}$ is a force. Notice that wrenches are not Lie algebra elements but elements of the vector space dual to the Lie algebra. Usually the force due to a potential is given by its gradient. The same is true here, in terms of the exterior derivative d, $\mathcal{W} = -d\Phi$. Pairing the wrench with an arbitrary screw S gives,

$$\mathcal{W}(S) = -d\Phi(S) = -\partial_S \Phi,$$

see [13, §. 4.20] for example. These calculations have already been done in section 2 above, the wrench is given by,

$$\mathcal{W} = \begin{pmatrix} \sum_i \lambda_i \mathbf{a}_i \times (\mathbf{a}_i - R\mathbf{b}_i - \mathbf{t}) \\ \sum_i \lambda_i (\mathbf{a}_i - R\mathbf{b}_i - \mathbf{t}) \end{pmatrix},$$

this could, of course, also have been deduced from elementary mechanics.

For the spring systems of section 2 an important object is the stiffness matrix of the system. In this section the stiffness matrix is computed by taking the second derivatives of the potential function.

An infinitesimal displacement of the body is represented by a screw. The wrench produced by a displacement $\mathbf{s}$ is given by $\mathcal{W} = K\mathbf{s}$, where K is the stiffness matrix.

The stiffness matrix is the hessian of the potential function, that is its matrix of second order partial derivatives, see [14, Ch 5]. This is only valid at an equilibrium position.

There have been attempts in the Robotics literature to extend these ideas to non-equilibrium configurations, see for example Griffis and Duffy [15] and Žefran and Kumar [16]. In this work, however, the classical definition of the stiffness matrix will be used.

Now write the result above for the wrench as

$$\mathcal{W} = \sum_i \lambda_i \begin{pmatrix} A_i & 0 \\ I_3 & 0 \end{pmatrix} \left(\tilde{\mathbf{a}}_i - M\tilde{\mathbf{b}}_i\right).$$

Differentiating this along an arbitrary screw gives,

$$\partial_S \mathcal{W} = \sum_i \lambda_i \begin{pmatrix} A_i & 0 \\ I_3 & 0 \end{pmatrix} \begin{pmatrix} RB_iR^T + T & I_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix},$$

using the result of section 2. Hence, the stiffness matrix is,

$$K = \begin{pmatrix} \sum_i \lambda_i (A_i RB_i R^T + A_i T) & \sum_i \lambda_i A_i \\ \sum_i \lambda_i (RB_i R^T + T) & \sum_i \lambda_i I_3 \end{pmatrix}.$$

This time choose the origin to be at the point $\sum_i \lambda_i \mathbf{a}_i$ and then use the equilibrium condition 2 to simplify the stiffness matrix to,

$$K = \begin{pmatrix} \sum_i \lambda_i A_i R B_i R^T & 0 \\ & \\ 0 & \sum_i \lambda_i I_3 \end{pmatrix}.$$

This is a particularly neat result but it is a little surprising at first sight. The term $\sum_i \lambda_i I_3$ in the bottom right corner means that the system has the same stiffness in any direction, irrespective of the stiffness of the individual springs and their arrangement. This is due to the assumption that the springs have zero natural length.

Next the problem of finding the index of the critical points found is reviewed. For brevity only the question, which of the critical points is a minimum of the potential energy, that is a stable equilibrium will be considered. That is: which of the solutions for R gives a stiffness matrix K with all positive eigenvalues? Notice that three of the eigenvalues of K are simply $\sum_i \lambda_i$, and this is positive if the spring constants λ_i are all positive. So it is only necessary to study the eigenvalues of the top-lefthand bock of K. After a little manipulation this can be written in terms of the matrix P or its polar decomposition,

$$\sum_i \lambda_i A_i R B_i R^T = RP^T + \mathrm{Tr}(RP^T) I_3 = R_i Q + \mathrm{Tr}(R_i Q) I_3.$$

Here Tr() represents the trace of the matrix. Now this matrix has the same eigenvectors as $R_i Q$ and hence as Q, remembering that R_i was either the identity or a rotation of π about an eigenvector of Q. So assume that the eigenvalues of Q are μ_1, μ_2 and μ_3 with corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$. Assuming that R_i is a rotation about eigenvector $\mathbf{e}_i$, the eigenvalues of the matrices can be found by considering the action on the eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. The eigenvalues for $R_0Q + \mathrm{Tr}(R_0Q)I_3$ are,

$$(2\mu_1 + \mu_2 + \mu_3), \qquad (\mu_1 + 2\mu_2 + \mu_3), \quad \text{and} \quad (\mu_1 + \mu_2 + 2\mu_3).$$

Only one other matrix needs to be considered since the others are just cyclic permutations, the eigenvalues of $R_1Q + \mathrm{Tr}(R_1Q)I_3$ are,

$$(2\mu_1 - \mu_2 - \mu_3), \qquad (\mu_1 - 2\mu_2 - \mu_3), \quad \text{and} \quad (\mu_1 - \mu_2 - 2\mu_3).$$

Now there are just two cases to consider, if $\det(P) > 0$ then Q is positive-definite and thus so are all its eigenvalues. In this case it is easy to see that the critical point represented by R_0 will be the minimum. This is the solution $R = R_p$.

In the other case $\det(P) < 0$, the classical polar decomposition gives us a positive-definite symmetric matrix and a reflection. Multiplying these by -1 gives a rotation and a negative-definite symmetric matrix. That is Q has all negative eigenvalues. Assume that these eigenvalues have the ordering $0 > \mu_1 \geq \mu_2 \geq \mu_3$, that is μ_1 is the eigenvalue of smallest magnitude, then the matrix, $R_1Q + \mathrm{Tr}(R_1Q)I_3$ will have all positive eigenvalues and hence R_1 corresponds to the minimum of the potential. So in general, if $\det(P) < 0$ the minimum solution is given by $R = R_pR_i$, where R_i is a rotation of π about the eigenvector of Q with eigenvalue of smallest magnitude.

6 Conclusions

The concept unifying the three problems studied in this work is the idea of functions defined on the group of rigid body transformations. It has been possible to find the stationary points of the functions and classify these critical points. This involves a simple technique of differentiating along a screw.

The results agree with those of Kanatani [11, Chap. 5], his methods were, perhaps, a little more elegant than the above. The methods used here are more general, all the critical points of the potential function have been found, not just the minimum and with a little more effort the index of each of them could have been found.

These ideas are central to the subject of Morse theory, a field of study which relates the critical points of functions defined on a manifold to the topology of the manifold itself. In this case the topology of the manifold, the underlying manifold of the group $SO(3)$, is well known, and this can be used to say something about the critical point of the potential function.

The problem of the springs is a slightly artificial one in that springs with zero natural length have been used. This simplifies the computations. It is possible to repeat most of the above analysis using springs with a finite natural length, see [17]. The number of critical points now becomes a very hard problem but will still be constrained by the topology of the group.

The problem of estimating a rigid motion from point data is reasonably realistic. The utility of the estimate will depend on the distribution of errors for the measured points. There are some results on this in the literature, see [18]. There seems to be much scope for further research in this direction.

There has only been space here to take a brief look at the implications for the Stewart platform. Certainly using the technique of differentiating along a screw the acceleration properties of the Stewart platform could be calculated. However, the usual difficulties on the geometric definition of a second derivative arise here. In particular circumstances it is clear what should be done, for example it is not too difficult to find the dynamics of a Stewart platform, see [19]. Again this appears to be a fertile area for future developments.

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