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Discrete Mathematics 296 (2005) 117–120

DISCRETE MATHEMATICS

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Note

Covering radii are not matroid invariants

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Received 25 September 2003; accepted 1 March 2005
Available online 12 May 2005

Abstract

We show by example that the covering radius of a binary linear code is not generally determined by the Tutte polynomial of the matroid. This answers Problem 361 (P.J. Cameron (Ed.), Research problems, Discrete Math. 231 (2001) 469–478).

Keywords: Linear code; Vector matroid; Tutte polynomial; Covering radius

The celebrated “Critical Theorem” by Crapo and Rota [2] shows that many detailed properties of a linear code $C \subseteq \mathbb{F}^E$ (over a field $\mathbb{F}$, with coordinates labeled by the elements of a set E) are determined by the associated vector matroid M_C. Greene [3] further demonstrated that the Tutte polynomial

$$T(M_C; x, y) = \sum_{A \subseteq E} (x-1)^{\rho_{M_C}(E)-\rho_{M_C}(A)} (y-1)^{|A|-\rho_{M_C}(A)}$$

often suffices to determine properties of C. Examples of such properties include the code length, dimension, minimum distance, and the weight enumerator. The purpose of this note is to present general code properties that are *not* determined by the Tutte polynomial of the associated matroid.

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doi:10.1016/j.disc.2005.03.002

The covering radius $r(C)$ of a code $C \subseteq \mathbb{F}^E$ is the maximal distance from C to any vector of $\mathbb{F}^E$. Equivalently, it is one less than the cardinality of the weight distribution, $a_0, a_1, \ldots, a_r$, of coset leaders v in the cosets $v + C$. The matrices

$$\begin{pmatrix} 1&0&0&0&0&0&1&1&0&1&0\\ 0&1&0&0&0&0&1&1&1&1&1\\ 0&0&1&0&0&0&1&0&1&1&1\\ 0&0&0&1&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0&0&0&1 \end{pmatrix} \quad \text{and}$$
$$\begin{pmatrix} 1&0&0&0&0&0&1&1&0&1&0\\ 0&1&0&0&0&0&1&1&1&0&0\\ 0&0&1&0&0&0&1&0&1&0&0\\ 0&0&0&1&0&0&1&0&1&1&1\\ 0&0&0&0&1&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0&0&0&1 \end{pmatrix}$$

over GF(2) represent matroids with a common Tutte polynomial given by

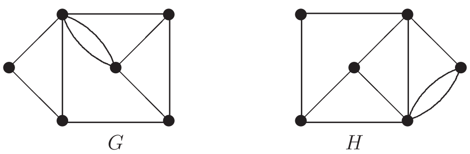
$$y^5 + (x+5)y^4 + (x^2+7x+12)y^3 + (x^3+8x^2+22x+15)y^2 + (2x^4+11x^3+27x^2+27x+7)y + (x^6+5x^5+13x^4+21x^3+19x^2+7x),$$

but generate a pair of codes with covering radii 2 and 3, respectively. Thus,

Theorem 1. *The covering radius $r(C)$ of a binary linear code C is not generally determined by the Tutte polynomial $T(M_C; x, y)$.*

Theorem 1 answers in the negative the question posed in [1, Problem 361].

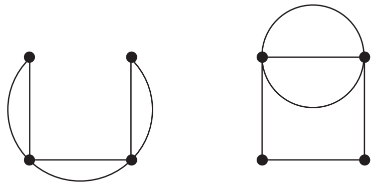
Gray (see [6]) observed that the pair of graphs presented below share a common Tutte polynomial, even though they are not 2-isomorphic.



Let $C(G)$ and $C(H)$ be the bond codes of G and H, i.e. the binary linear codes spanned by the characteristic vectors of the bonds of G and H, respectively. The non-isomorphic matroids $M_{C(G)} = M(G)$ and $M_{C(H)} = M(H)$ have in common their Tutte polynomial. Furthermore, the codes $C(G)$ and $C(H)$ both have covering radius 3. However, the weight distributions of their coset leaders are 1, 9, 20, 2 and 1, 9, 18, 4, respectively. Therefore,

Theorem 2. *The weight distribution of coset leaders of a bond code C is not, in general, determined by the Tutte polynomial* $T(M_C; x, y)$ *together with the covering radius* $r(C)$.

Consider the following graphs:



G H

The bond codes $C(G)$ and $C(H)$ both have weight enumerator

$$x^6 + 3x^4y^2 + 3x^2y^4 + y^6,$$

but they have covering radii 3 and 2, respectively. It follows that

Theorem 3. *The covering radius* $r(C)$ *of a bond code C is not, in general, determined by the weight enumerator of C.*

To conclude, consider the general case in which $C \subseteq \mathbb{F}^E$ is a linear code and M_C is not necessarily uniquely representable. For instance, the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 4 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 6 & 5 \end{pmatrix}$$

are (inequivalent) representations of the uniform matroid $U_{3,6}$ over $\mathrm{GF}(7)$. The codes generated (over $\mathrm{GF}(7)$) by each matrix have covering radii 2 and 3, respectively. This illustrates the following result of A.N. Skorobogatov [7].

Theorem 4. *The covering radius* $r(C)$ *of a linear code* $C \subseteq \mathbb{F}^E$ *is not generally determined by the associated matroid* M_C.

Acknowledgements

All computations were conducted with the aid of GAP [4] and MAGMA [5].

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