Centrodes and Lie algebra

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Abstract—*The classical subject of planar kinematics is reviewed in the setting of Lie algebra and differential geometry. In particular, the classical centrode curves of a rigid motion are related to the derivatives in the Lie algebra. Several examples of finding centrode curves in different situations are given. The case where a rigid motion is determined by one curve rolling on another is studied in some detail.*

Keywords: Lie algebra, fixed centrode, moving centrode, planar kinematics.

I. Introduction

In this work the classical subject of planar kinematics is revisited using techniques from Lie algebra. The motivation behind this reexamination of old work is twofold, first it is hoped that this will give a simple account of this subject including efficient computational tools. Secondly, by putting planar kinematics in a general geometrical form it is hoped that spatial kinematics can be treated in a similar fashion. The central result is the close connection between the centrode curve and the derivative in the Lie algebra. The derivative in the fixed frame can be projected to the fixed centrode curve while the derivative in the moving frame projects to the moving centrode.

Several example are given showing how to compute the centrode curves from the derivatives in different circumstances. Special attention is paid to planar motions generated by rolling on curve on another.

To begin a brief review of the group of planar motions $SE(2)$, is given, more details on this approach can be found in [5].

II. The Group of Planar Motions

As usual we represent the group of planar motions by 3×3 matrices of the form,

$$\begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where in the partitioned form R represents the 2×2 rotation matrix and $\mathbf{t}$ a translation vector. The action of such a transformation on a point is given by the matrix product,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \tag{2}$$

or in partitioned form,

$$\begin{pmatrix} \mathbf{p}' \\ 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix}. \tag{3}$$

In the above we have implicitly assumed that the transformations were active, that is they moved the points in the plane. It is also necessary to consider passive transformations. Such transformations are changes of coordinates. Suppose that,

$$M = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \tag{4}$$

is an active transformation that transforms a coordinate frame Σ into a new frame ${}^n\Sigma$, then a point $\mathbf{p}$ in the first frame will have coordinates in the second frame ${}^n\mathbf{p}$ given by,

$$\begin{pmatrix} {}^n\mathbf{p} \\ 1 \end{pmatrix} = M^{-1} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R^T & -R^T\mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{5}$$

Further, suppose that N is an active transformation written in the coordinates of the first frame, in the new frame the components of the matrix will be,

$${}^nN = M^{-1}NM, \tag{6}$$

where as above M is the active transformation which takes the first frame to the second, written in the coordinates of the first frame.

A general planar transformation is either a rotation about a fixed point or exceptionally a pure translation. Notice that a rotation about a point $\mathbf{q}$ is given by the conjugation,

$$\begin{pmatrix} I & \mathbf{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & \mathbf{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & -\mathbf{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & (I-R)\mathbf{q} \\ 0 & 1 \end{pmatrix}, \tag{7}$$

where I is the 2×2 identity matrix. The statement that all planar transformations are rotations about a fixed point amounts to stating that we can solve the equation,

$$(I - R)\mathbf{q} = \mathbf{t} \tag{8}$$

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for $\mathbf{q}$, the fixed point or centre of rotation, given any R and $\mathbf{t}$. The exception occurs when $R = I$ and hence the transformation is a pure translation. Clearly rotations about a fixed point and translations along a fixed direction are the one-parameter subgroups of the planar motions group.

A. The Lie algebra and the Exponential Map

To find the elements of the Lie algebra to group we can differentiate the one parameter subgroups at the identity. Differentiating an arbitrary rotation gives,

$$\frac{d}{d\theta}\begin{pmatrix}\cos\theta & -\sin\theta & (1-\cos\theta)q_x+\sin\theta q_y \\ \sin\theta & \cos\theta & -\sin\theta q_x+(1-\cos\theta)q_y \\ 0 & 0 & 1\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix}0 & -1 & q_y \\ 1 & 0 & -q_x \\ 0 & 0 & 0\end{pmatrix}. \quad (9)$$

For the translations we get,

$$\frac{d}{d\phi}\begin{pmatrix}1 & 0 & \phi t_x \\ 0 & 1 & \phi t_y \\ 0 & 0 & 1\end{pmatrix}\Bigg|_{\phi=0} = \begin{pmatrix}0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 0\end{pmatrix}. \quad (10)$$

If we write $E = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$ then we can write the Lie algebra element corresponding to a rotation about $\mathbf{q}$ in partitioned form as,

$$L = \begin{pmatrix}-E & E\mathbf{q} \\ 0 & 0\end{pmatrix} \quad (11)$$

Notice also that these matrices satisfy the cubic equation,

$$L^3 + L = 0. \quad (12)$$

So it is easy to see that the exponential of such a matrix obeys the Rodrigues formula familiar from the rotation group $SO(3)$, since the elements of the Lie algebra satisfy the same equation. That is,

$$e^{\theta L} = I + \sin\theta L + (1-\cos\theta)L^2. \quad (13)$$

This time I is the 3×3 identity matrix.

The exponential of a pure translation is much simpler, let,

$$T = \begin{pmatrix}0 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 0\end{pmatrix}. \quad (14)$$

Then,

$$e^{\phi T} = I + \phi T, \quad (15)$$

since clearly $T^2 = 0$.

B. Derivatives

The derivatives of one-parameter subgroups are also simple. Suppose L is a fixed element of the Lie algebra and θ is a scalar function of a parameter t, which might represent time. Now we have that,

$$\frac{d}{dt}e^{\theta L} = \dot{\theta}Le^{\theta L} \quad (16)$$

However, if L is also a function of t then things are not so simple. The problem is that $\dot{L}$ may not commute with L and so if we were to differentiate the infinite series for the exponential term by term we have to deal with terms such as,

$$\frac{d}{dt}L^2 = \dot{L}L + L\dot{L}. \quad (17)$$

This problem was solved by Hausdorff in [3]. He showed that in any Lie group,

$$\left(\frac{d}{dt}e^X\right)e^{-X} = \sum_{i=0}^{\infty}\frac{1}{(i+1)!}\mathrm{ad}_X^i(\dot{X}), \quad (18)$$

where $\mathrm{ad}_A^i(C)$ denotes the iterated commutator; $\mathrm{ad}_A^0(C) = C$, $\mathrm{ad}_A^1(C) = [A, C]$, $\mathrm{ad}_A^2(C) = [A, [A, C]]$ and so forth. Clearly the above expression is an element of the Lie algebra and we will denote it $X_d = d/dt(e^X)e^{-X}$. We also have,

$$e^{-X}\left(\frac{d}{dt}e^X\right) = \sum_{i=0}^{\infty}\frac{(-1)^i}{(i+1)!}\mathrm{ad}_X^i(\dot{X}), \quad (19)$$

and this will be denoted X_b. Clearly X_b and X_d are related,

$$X_b = e^{-X}X_d e^X \quad (20)$$

So if X_d is the derivative with respect to a global fixed frame then X_b is the derivative expressed in the frame moving with the body.

In the planar case we can compute these derivatives explicitly using the Rodrigues formula given in the previous section, if

$$L = L(t) = \begin{pmatrix}-E & E\mathbf{r}(t) \\ 0 & 0\end{pmatrix} \quad (21)$$

then we have,

$$\begin{aligned} L_d &= \frac{d}{dt}(e^{\theta L})\,e^{-\theta L} = \dot{\theta}\begin{pmatrix}-E & E\mathbf{r} \\ 0 & 0\end{pmatrix} \\ &\quad + \sin\theta\begin{pmatrix}0 & E\dot{\mathbf{r}} \\ 0 & 0\end{pmatrix} + (1-\cos\theta)\begin{pmatrix}0 & \dot{\mathbf{r}} \\ 0 & 0\end{pmatrix} \end{aligned} \quad (22)$$

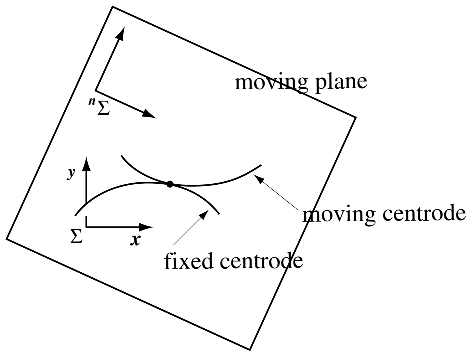

Fig. 1. The Fixed and Moving Planes.

III. Centrodes

Imagine a plane or lamina (called the moving plane) moving relative to a fixed plane. The pole of the motion is usually defined as the point in the plane that is instantaneously at rest, see [4, Ch.5] or [1, Ch. VIII] for example. As the motion proceeds the pole traces out a curve in the fixed plane, called the fixed centrode and also a curve in the moving plane—the moving centrode. At any instant the moving centrode and fixed centrode will meet at the pole. It is sometimes useful to imagine the motion as being produced by rolling the moving centrode on the fixed centrode, see figure 1, this will also be studied in more detail in section IV below.

In this section it will be shown how the fixed and moving centrodes can be derived from the description of the motion as an exponential. Hence assume that the motion of the moving plane relative to the fixed one is given by,

$$M(t) = e^{\theta L} \tag{23}$$

The parameter t here could represent time, in which case we can talk about velocities and accelerations of points, but in general it is simply a parameter. However we expect that both θ and $L = \begin{pmatrix} -E & E\mathbf{r} \\ 0 & 0 \end{pmatrix}$ are functions of t: $\theta = \theta(t), \quad L = L(t)$.

A. The Fixed Centrode

We can find the pole of the motion by setting the velocity of an arbitrary point to zero and then solving for the coordinates of the point.

Consider a point in the moving lamina with coordinates (relative to the fixed frame) $\mathbf{p}_0$ when $t = 0$. At any subsequent time we can find the coordinates of the point from the following,

$$\begin{pmatrix} \mathbf{p}(t) \\ 1 \end{pmatrix} = M(t) \begin{pmatrix} \mathbf{p}_0 \\ 1 \end{pmatrix} = e^{\theta L} \begin{pmatrix} \mathbf{p}_0 \\ 1 \end{pmatrix}. \tag{24}$$

So the velocity of the point can be found by differentiating,

$$\begin{pmatrix} \dot{\mathbf{p}} \\ 0 \end{pmatrix} = L_d e^{\theta L} \begin{pmatrix} \mathbf{p}_0 \\ 1 \end{pmatrix} = L_d \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix}. \tag{25}$$

and the equation for the fixed centrode is obtained by setting this to zero,

$$L_d \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \mathbf{0}. \tag{26}$$

Now let us assume that the derivative L_d has the form,

$$L_d = \omega \begin{pmatrix} -E & E\mathbf{c}_f \\ 0 & 0 \end{pmatrix}. \tag{27}$$

Setting the velocity of the point to zero we find that the point on the fixed centrode satisfies,

$$\omega(\mathbf{p} - \mathbf{c}_f) = \mathbf{0}. \tag{28}$$

So long as $\omega \neq 0$ we see that the fixed centrode curve is given by $\mathbf{c}_f$. From (22) above we see that in terms of curve in the Lie algebra we have $\omega = \dot{\theta}$ and

$$\mathbf{c}_f = \mathbf{r} + \frac{1}{\dot{\theta}} \begin{pmatrix} \sin\theta & \cos\theta - 1 \\ 1 - \cos\theta & \sin\theta \end{pmatrix} \dot{\mathbf{r}}. \tag{29}$$

Notice that equation (27) essentially gives a projection from the Lie algebra element L_d to the classical fixed centrode curve $\mathbf{c}_f$. Care must be taken if $\dot{\theta} = 0$, this can happen if the motion is instantaneously a translation, in such cases L_d exist even though $\mathbf{c}_f$ is not defined.

B. The Moving Centrode

To find the moving centrode we can repeat the argument above in the moving plane to find the point instantaneously at rest in the moving plane. However we can also simply transfer the fixed centrode into the moving coordinates. Suppose that $\mathbf{p}$ is a point on the fixed centrode, in the moving frame the coordinates will be,

$$\begin{pmatrix} {}^n\mathbf{p} \\ 1 \end{pmatrix} = e^{-\theta L} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix}. \tag{30}$$

In these coordinates the equation for the centrode, (26) above, becomes,

$$e^{-\theta L} L_d e^{\theta L} \begin{pmatrix} {}^n\mathbf{p} \\ 1 \end{pmatrix} = L_b \begin{pmatrix} {}^n\mathbf{p} \\ 1 \end{pmatrix} = \mathbf{0}. \tag{31}$$

So the moving centrode is effectively the derivative in the moving frame. This derivative can be computed in terms of the curve in the Lie algebra which generates the motion as in (22) above:

$$\begin{aligned} L_b &= e^{-\theta L} \frac{d}{dt}(e^{\theta L}) = \dot{\theta} \begin{pmatrix} -E & E\mathbf{r} \\ 0 & 0 \end{pmatrix} \\ &\quad + \sin\theta \begin{pmatrix} 0 & E\dot{\mathbf{r}} \\ 0 & 0 \end{pmatrix} - (1 - \cos\theta) \begin{pmatrix} 0 & \dot{\mathbf{r}} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{32}$$

Assuming that,

$$L_b = \varpi \begin{pmatrix} -E & E\mathbf{c}_m \\ 0 & 0 \end{pmatrix}, \tag{33}$$

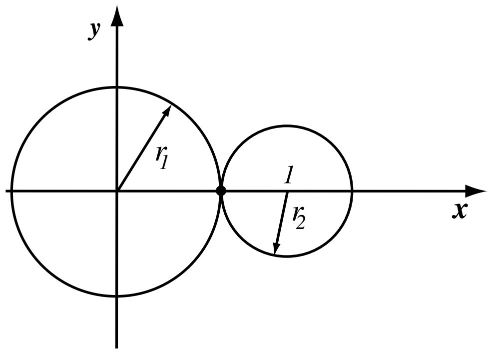

Fig. 2. Cycloidal Motion.

then it is straightforward to see that the moving centrode, expressed in the moving coordinates, will be,

$$\mathbf{c}_m = \mathbf{r} + \frac{1}{\dot\theta}\begin{pmatrix} \sin\theta & 1-\cos\theta \\ \cos\theta - 1 & \sin\theta \end{pmatrix}\dot{\mathbf{r}}. \tag{34}$$

C. Example—Cycloidal motion

The first example here is the motion generated by a circle rolling without slipping on another circle. We will assume that the circles have radii r_1 and r_2 with $r_1 + r_2 = 1$. The fixed circle will have the origin as its centre and at $t = 0$ the centre of the moving circle will be at the point $\mathbf{i} = (1,\,0)^T$, see figure 2. The motion of the moving plane, attached to the rolling circle, can be thought of as a composition of a rotation about the point $\mathbf{i}$ followed by a rotation about the origin. The two rotation angles are related by the condition that the circles roll on each other without slipping. Suppose that at some time t the angle turned about the origin is α and at the same time the rotation about $\mathbf{i}$ is β, then the common length rolled about the circumferences of the circles must be equal: $\alpha r_1 = \beta r_2$, so that,

$$\alpha = \lambda r_2, \qquad \beta = \lambda r_1, \tag{35}$$

for some constant λ. The motion can be written as a product of exponentials,

$$M(t) = e^{\alpha t L}e^{\beta t K}, \tag{36}$$

where $L = \begin{pmatrix} -E & \mathbf{0} \\ 0 & 1 \end{pmatrix}$ and $K = \begin{pmatrix} -E & E\mathbf{i} \\ 0 & 1 \end{pmatrix}$. Hence for the derivative we have,

$$\left(\frac{d}{dt}M\right)M^{-1} = \alpha L + \beta e^{\alpha t L}Ke^{-\alpha t L}. \tag{37}$$

A short calculation gives the derivative as,

$$\omega\begin{pmatrix} -E & E\mathbf{c}_f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(\alpha+\beta) & \beta\sin\alpha t \\ (\alpha+\beta) & 0 & -\beta\cos\alpha t \\ 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

Hence the fixed centrode is given by,

$$\mathbf{c}_f = \frac{\beta}{\alpha+\beta}\begin{pmatrix} \cos\alpha t \\ \sin\alpha t \end{pmatrix} = \begin{pmatrix} r_1\cos\alpha t \\ r_1\sin\alpha t \end{pmatrix}, \tag{39}$$

which is just the equation for the fixed circle. The moving centrode is only a little harder to compute, the result is,

$$\mathbf{c}_m = \begin{pmatrix} 1 - r_2\cos\beta t \\ r_2\sin\beta t \end{pmatrix}, \tag{40}$$

this is, of course, the parameterisation of the moving circle.

D. Example—Cardan Motion

In this motion two points on the moving plane are constrained to lie on a pair of (non-parallel) lines in the fixed plane, see [1, Ch. 9 §11] for example. Again the motion can be parameterised as a product of two exponentials, for simplicity here it will be assumed that the fixed lines are the x and y axes, so if we begin with one point at the origin and the other point one unit along the x-axis, any motion consistent with the constraints can be realised as a rotation about the origin followed by a translation along the y-axis:

$$M(t) = e^{-\sin\theta J}e^{\theta L}, \tag{41}$$

where L is as in the previous section and $J = \begin{pmatrix} 0 & \mathbf{j} \\ 0 & 0 \end{pmatrix}$ with $\mathbf{j} = (0,\,1)^T$. Notice that the two exponentials depend on each other, this relation is easily found from elementary trigonometry. The derivative in the fixed plane is,

$$\left(\frac{d}{dt}M\right)M^{-1} = -\dot\theta\cos\theta J + \dot\theta e^{-\sin\theta J}Le^{\sin\theta J}. \tag{42}$$

Evaluating this gives,

$$\left(\frac{d}{dt}M\right)M^{-1} = \dot\theta\begin{pmatrix} 0 & -1 & -\sin\theta \\ 1 & 0 & -\cos\theta \\ 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

and hence the fixed centrode is given by,

$$\mathbf{c}_f = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}. \tag{44}$$

So the fixed centrode is a unit circle about the origin.

The derivative in the moving plane is given by,

$$M^{-1}\left(\frac{d}{dt}M\right) = -\dot\theta\cos\theta e^{-\theta L}Je^{\theta L} + \dot\theta L. \tag{45}$$

Evaluating this expression gives,

$$M^{-1}\left(\frac{d}{dt}M\right) = \dot\theta\begin{pmatrix} 0 & -1 & -\cos\theta\sin\theta \\ 1 & 0 & -\cos^2\theta \\ 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

Using trigonometric identities this gives the moving centrode (at $t = 0$) as,

$$\mathbf{c}_m = \begin{pmatrix} (1/2) + (1/2)\cos 2\theta \\ -(1/2)\sin 2\theta \end{pmatrix}. \tag{47}$$

That is, the moving centrode is also a circle, of radius $1/2$ and centred at the point $(1/2,\, 0)$ when $t = 0$. So in fact this motion is also a cycloidal motion.

IV. Rolling Curves

In this section the motion generated by one curve rolling on another will be considered in a little more detail. Let us assume that the curve in the fixed plane is given by,

$$\mathbf{f}(t) = \begin{pmatrix} \mathbf{p}_f(t) \\ 1 \end{pmatrix} = \begin{pmatrix} x_f(t) \\ y_f(t) \\ 1 \end{pmatrix}. \tag{48}$$

In the moving plane we have a curve given by,

$$\mathbf{m}(t) = \begin{pmatrix} \mathbf{p}_m(t) \\ 1 \end{pmatrix} = \begin{pmatrix} x_m(t) \\ y_m(t) \\ 1 \end{pmatrix}. \tag{49}$$

The same parameter t, has been used here because we assume that the curves have been parameterised in such a way that at parameter value t, the curves will meet at the point given by $\mathbf{f}(t)$ on one curve and $\mathbf{m}(t)$ on the other. So the rigid-body motion must satisfy,

$$\mathbf{f}(t) = e^{\theta L}\mathbf{m}(t). \tag{50}$$

To ensure rolling the tangents to the curve must be parallel at the contact point. The tangents to the curves are,

$$\dot{\mathbf{f}}(t) = \frac{d}{dt}\mathbf{f}(t) \quad \text{and} \quad \dot{\mathbf{m}}(t) = \frac{d}{dt}\mathbf{m}(t).$$

The requirement that these vectors should be parallel can be expressed as,

$$\dot{\mathbf{f}}(t)^T \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} e^{\theta L}\dot{\mathbf{m}}(t) = 0. \tag{51}$$

This is essentially a 2×2 determinant written using the anti-symmetric matrix E introduced above. Notice that slipping has not been excluded here, this will be considered in a moment.

Now if we simply differentiate equation (50) we get,

$$\dot{\mathbf{f}}(t) = L_d e^{\theta L}\mathbf{m}(t) + e^{\theta L}\dot{\mathbf{m}}(t) = L_d\mathbf{f}(t) + e^{\theta L}\dot{\mathbf{m}}(t). \tag{52}$$

This equation can be multiplied by $\dot{\mathbf{f}}(t)^T \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ to give,

$$0 = \dot{\mathbf{f}}(t)^T \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} L_d\mathbf{f}(t), \tag{53}$$

where the anti-symmetry of E and equation (51) have been used to simplify the result. Next we can use the partitioned form of the curves given in (48) and (49) and the fact that $E^2 = -I$ to give,

$$\omega\dot{\mathbf{p}}_f \cdot (\mathbf{p}_f - \mathbf{c}_f) = 0. \tag{54}$$

When $\omega \neq 0$, this implies that the pole of the motion $\mathbf{c}_f$, must lies along the normal to the fixed curve $\mathbf{p}_f$.

To be more precise we must give a relationship between the tangents of the two curves, suppose that,

$$(1 + \gamma(t))\dot{\mathbf{f}}(t) = e^{\theta L}\dot{\mathbf{m}}(t), \tag{55}$$

where $\gamma(t)$ is a scalar function of the parameter t. The idea is that $\gamma(t)$ is a measure of slip, we have that,

$$\gamma(t) = (|\dot{\mathbf{p}}_m| - |\dot{\mathbf{p}}_f|)/|\dot{\mathbf{p}}_f|. \tag{56}$$

So if there is no slipping $\gamma(t) = 0$.

Now returning to equation (52) we get,

$$\gamma(t)\dot{\mathbf{f}}(t) + L_d\mathbf{f}(t) = 0. \tag{57}$$

Extracting the planar part of this equation, multiplying by E and rearranging give an equation for the fixed centrode in terms of the fixed curve,

$$\mathbf{c}_f = \mathbf{p}_f + \frac{\gamma}{\omega}E\dot{\mathbf{p}}_f, \tag{58}$$

where explicit dependance on t has been suppressed for clarity. Clearly we could produce a similar expression for the moving centrode in terms of the moving curve,

$$\mathbf{c}_m = \mathbf{p}_m + \frac{\gamma}{\omega(1+\gamma)}E\dot{\mathbf{p}}_m. \tag{59}$$

Finally here, suppose that we require the moving curve to roll along the fixed curve without slipping. This implies that the arc-lengths of the curves must agree, the differential form of this constraint is that the tangent vectors must be not only parallel but must also have the same length, that is,

$$\dot{\mathbf{f}}(t) = e^{\theta L}\dot{\mathbf{m}}(t). \tag{60}$$

In other words the function $\gamma(t) = 0$. Substituting this in equation (58) we have that,

$$\mathbf{c}_f = \mathbf{p}_f, \tag{61}$$

so long as $\omega \neq 0$. This means that we can only reproduce the given motions by rolling one curve on another without slipping if the curves are the centrode curves. This gives another characterisation of the centrode curves. Notice also that these simple observations make the computations done in section III-C unnecessary.

Equation (58) above gives a simple method for computing the fixed centrode given the two curves which roll on each other, where γ is computed from equation (56).

Alternatively, if the motion, and hence its fixed centrode, is known then a fixed curve may be chosen. The fixed curve must satisfy the consistency condition (54). The moving curve is then found from equation (50).

A. Gears Teeth Profiles

An important application of the above is to the shape of gear teeth. This subject has a long and successful history. However, with the geometric methods developed here we can give very concise derivations of some of the classical results.

In standard approaches to this subject the pivots of the gear wheels are considered fixed and both wheels move. Here we will consider one of the gears to be fixed, that is to say we work in a coordinate frame fixed with respect to the gear. The desired motion of the pinion is then clearly the cycloidal motion studied in section III-C above. Moreover, the fixed and moving centrodes of this motion have already been found above. The fixed and moving curves which roll on each other to produce the motion correspond to the profiles of the gear teeth.

In this application the fixed centrode is known to be a circle which can be written in parameterised form as,

$$\mathbf{c}_f = \begin{pmatrix} r_1 \cos \alpha t \\ r_1 \sin \alpha t \end{pmatrix}. \tag{62}$$

The most common (if not universal) choice of profile for gear teeth is the involute of a circle. This curve can be constructed by unwinding a taut string coiled around the circle. More details on this and other geometric constructions for plane curves can be found in [2]. A parameterisation of the involute consistent with the parameterisation of the circle given above is,

$$\mathbf{p}_f = \begin{pmatrix} r_1 \cos \alpha t + \alpha r_1 t \sin \alpha t \\ r_1 \sin \alpha t - \alpha r_1 t \cos \alpha t \end{pmatrix}. \tag{63}$$

The derivative of this curve is then,

$$\dot{\mathbf{p}}_f = \begin{pmatrix} \alpha^2 r_1 t \cos \alpha t \\ \alpha^2 r_1 t \sin \alpha t \end{pmatrix}, \tag{64}$$

and clearly this satisfies the consistency condition (54). Using equation (50), and the relations $r_1 + r_2 = 1$ and $\alpha r_1 = \beta r_2$, the parameterisation of the moving curve is,

$$\mathbf{p}_m = \begin{pmatrix} 1 - r_2 \cos \beta t - \beta r_2 t \sin \beta t \\ r_2 \sin \beta t - \beta r_2 t \cos \beta t \end{pmatrix}, \tag{65}$$

which is, of course, a involute curve of the moving centrode.

V. The Planar 4-Bar Mechanism

In this final section we take a brief look at yet another application. The coupler bar in a planar 4-bar mechanism executes a sequence of planar motions so the problem here is to find the centrode curves of this motion. A complete solution to this problem is beyond the scope of this work, but a method will be sketched which solves the problem.

Imagine the that moving plane is fixed to the coupler bar of the mechanism. As the mechanism moves two points in the moving plane lie on circles in the fixed plane. Assume that the two points are $\mathbf{p}_1 = (x_1, y_1, 1)^T$ and $\mathbf{p}_2 = (x_2, y_2, 1)^T$. If the lengths of the bars are l_1, l_2, l_3 and l_4, then the circles will be,

$$\begin{aligned} x_1^2 + y_1^2 &= l_1^2, & (66) \\ (x_2 - l_4)^2 + y_2^2 &= l_3^2. & (67) \end{aligned}$$

These equations together with the equation expressing the fact that $\mathbf{p}_1$ and $\mathbf{p}_2$ are separated by a distance l_2,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2, \tag{68}$$

determine the kinematics of the mechanism. To find the fixed centrode of this motion suppose that there is a fixed 'home' position for the mechanism so that at $t = 0$ we have $\mathbf{p}_1(0) = \mathbf{p}_1^0$ and also $\mathbf{p}_2(0) = \mathbf{p}_2^0$. So that subsequent positions of these points are given by, $\mathbf{p}_1(t) = e^{\theta L}\mathbf{p}_1^0$ and $\mathbf{p}_2(t) = e^{\theta L}\mathbf{p}_2^0$. Now the two equations (66) and (67) above can be written in matrix form as $\mathbf{p}_1^T Q_1 \mathbf{p}_1 = 0$ and $\mathbf{p}_2^T Q_2 \mathbf{p}_2 = 0$ where,

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -l_1^1 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1 & 0 & -l_4 \\ 0 & 1 & 0 \\ -l_4 & 0 & l_4^2 - l_3^2 \end{pmatrix}.$$

Substituting the exponential form for the points and differentiating with respect to t gives the equations,

$$\begin{aligned} \mathbf{p}_1^T (L_d^T Q_1 + Q_1 L_d)\mathbf{p}_1 &= 0, & (69) \\ \mathbf{p}_2^T (L_d^T Q_2 + Q_2 L_d)\mathbf{p}_2 &= 0. & (70) \end{aligned}$$

These equations are linear in the components of $\mathbf{c}_f$, the fixed centrode, but quadratic in x_1, x_2, y_1 and y_2. These four variables can be eliminated using the kinematic equations (66), (67) and (68).

VI. Conclusions

The subject of planar kinematics is well known, however equations (29) and (34), which give the centrodes in terms of the motion expressed as an exponential, seem to be new. The purpose of this work was not to find new results. Rather the idea was to derive familiar results in a new way. This new method is elegant and efficient, its key feature is that it is based on concepts from the theory of Lie algebra and differential geometry.

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