Displacement Varieties for Some PUP Linkages

J.M. Selig

London South Bank University, London SE1 0AA, U.K.
seligjm@lsbu.ac.uk

Abstract. In this work sub-varieties of the Study quadric of rigid-body displacements are investigated. The particular sub-varieties considered are the possible displacements which can be performed by the end-effector of certain PUP linkages. In general these varieties are linear projections of a 4-dimensional Segre variety in 15-dimensional projective space. The different linkages correspond to projections with different centres. For one particular linkage the intersection of three such sub-varieties is studied, this intersection corresponds to the assembly configurations of a parallel robot with 3 such PUP legs.

Keywords: Study quadric, Segre varieties, 3PUP manipulator

1 Introduction

In a series of papers Dai and co-workers introduced and studied several novel parallel manipulators consisting of 3 PUP legs, [4, 9, 1]. In this work the geometry of the individual legs are considered. That is, the linkages considered here are comprised of a serial chain with a prismatic joint, two revolute joints with their axes meeting orthogonally and a final prismatic joint. There are several types of such a linkage depending on the angles between directions of the prismatic joints and the revolute joints. Only a few examples are studied in this work, the types chosen are those actually used in the work of Dai *et al.*

The main object of study is the displacement variety of the linkage. This is the algebraic variety of possible rigid-body displacements that the end-effector of the linkage can perform. The model used for the rigid-body displacements will be the Study quadric, a non-singular quadric in $\mathbb{P}^7$. With homogeneous coordinate $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ for the 7-dimensional projective space, the equation of the Study quadric is,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0.$$

Points in this 6-dimensional quadric variety are in 1-to-1 correspondence with the set of rigid-body displacements of space, with the exception of the 3-dimensional plane $a_0 = a_1 = a_2 = a_3 = 0$. This 3-plane will be denoted A_∞ in the following and consists of a set of "ideal elements" of the group of rigid-body displacements. The displacement varieties considered will be sub-varieties of the Study quadric defined by homogeneous algebraic equations in the homogeneous coordinates of $\mathbb{P}^7$. For more details see [6].

2 Generalities

Expressing elements of the group of rigid displacements as dual quaternions, the displacements generated by prismatic or revolute joints are lines in the Study quadric. For example, rotations about the z-axis are represented as dual quaternions of the form $g = (\alpha + \beta k)$. Since the rotation axis passes through the origin here, there is no dual part to the element. The parameters α and β are arbitrary but cannot be zero simultaneously. Only the ratio of the parameters is important, so α and β can be considered as homogeneous parameters of a projective line $\mathbb{P}^1$. Dividing both α and β by a common factor does not change the group element, so they may be divided by $\sqrt{\alpha^2 + \beta^2}$ and hence these parameter can be considered as the cosine and sine of a rotation angle.

For a prismatic joint, say in the z direction, the corresponding $\mathbb{P}^1$ of group elements is given by, $g = \lambda + \varepsilon k\mu$. In this case the ratio μ/λ gives the distance translated along the joint.

The algebraic variety formed as a product of projective spaces is known as a Segre variety. A product of 4 lines, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ gives a 4-dimensional variety in $\mathbb{P}^{15}$. The standard map from the Segre variety to $\mathbb{P}^{15}$ is given by identifying the 16 homogeneous coordinates of $\mathbb{P}^{15}$ with the 16 possible monomials formed from the coordinates of the 4 $\mathbb{P}^1$s, one coordinate from each line. As a simple example consider $\mathbb{P}^1 \times \mathbb{P}^1$ where the coordinates of the lines are $(\alpha_1 : \beta_1)$ and $(\alpha_2 : \beta_2)$. The Segre mapping to $\mathbb{P}^3$ is then given by,

$$(\alpha_1 : \beta_1) \times (\alpha_2 : \beta_2) \longmapsto (\alpha_1\alpha_2 : \alpha_1\beta_2 : \beta_1\alpha_2 : \beta_1\beta_2).$$

The points in the image of the map, the points in the Segre variety, satisfy the single quadratic equation, $X_1X_4 - X_2X_3 = 0$, where X_i are the homogeneous coordinates of the $\mathbb{P}^3$. The case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is treated in some detail in [5].

There are several ways to find the degree of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, extending the methods used in[2], using homology or cohomology theory. A simple method that shows that the degree of an n-fold product of lines, $\left(\mathbb{P}^1\right)^n$ has degree $n!$ is given in [3]. Hence, the degree of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is 24. The variety also satisfies a large number of quadratic equations.

In this work the varieties are all linear projections of this Segre variety to $\mathbb{P}^7$, the different types of linkage are given by different projections, that is projections from different $\mathbb{P}^7$s in the $\mathbb{P}^{15}$. Note that this would also apply to general 4R serial linkages.

3 First Type

Figure 1 shows four different types of PUP linkage, each in a standard configuration. In this section the leftmost of these mechanisms will be considered. By choosing the origin of coordinate to be at the centre of the U-joint and with the two revolute joints of the U joint aligned with the x and y axes, the possible displacements that the linkage can perform may be parameterised as,

$$g = (\lambda_1 + \mu_1\varepsilon k)(\alpha_2 + \beta_2 i)(\alpha_3 + \beta_3 j)(\lambda_4 + \mu_4\varepsilon k)$$

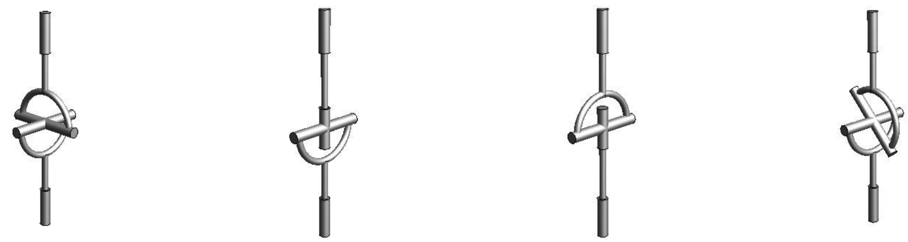

Fig. 1. Four different types of PUP linkage

Multiplying out this product gives,

$$g = a_0 + a_1 i + a_2 j + a_3 k + c_0 \varepsilon + c_1 \varepsilon i + c_2 \varepsilon j + c_3 \varepsilon k$$

where,

$$\begin{aligned} a_0 &= \lambda_1\lambda_4\alpha_2\alpha_3, & c_0 &= -(\lambda_1\mu_4 + \mu_1\lambda_4)\beta_2\beta_3, \\ a_1 &= \lambda_1\lambda_4\beta_2\alpha_3, & c_1 &= (\lambda_1\mu_4 - \mu_1\lambda_4)\alpha_2\beta_3, \\ a_2 &= \lambda_1\lambda_4\alpha_2\beta_3, & c_2 &= -(\lambda_1\mu_4 - \mu_1\lambda_4)\beta_2\alpha_3, \\ a_3 &= \lambda_1\lambda_4\beta_2\beta_3, & c_3 &= (\lambda_1\mu_4 + \mu_1\lambda_4)\alpha_2\alpha_3. \end{aligned}$$

By inspection these coordinates satisfy 3 linearly independent homogeneous quadratic equations,

$$\begin{aligned} 0 &= a_0 a_3 - a_1 a_2, \\ 0 &= a_0 c_0 + a_3 c_3, \\ 0 &= a_0 c_0 + a_1 c_1 + a_2 c_2 + a_3 c_3. \end{aligned}$$

The last equation here is just the equation of the Study quadric. Using a symbolic algebra computer package (Singular) it can be determined that the three equations generate a prime ideal. Hence the intersection of the three quadrics determined by the equations form an irreducible projective variety. Thus the degree of the variety is simply $2 \times 2 \times 2 = 8$.

Notice that the variety determined by the 3 quadric contains the 3-plane A_∞ of ideal displacements. This make it difficult to study structures composed of two or three such legs combined in parallel—the intersection of two or three such varieties will not form a complete intersection and hence Bézout's theorem cannot be applied. To investigate this we need to look at the general case for this type of linkage. Consider the PUP linkage illustrated in figure 2. To keep the computations as simple as possible the origin is still placed at the centre of the U joint with the x and y axes aligned with the first and second revolute joints. The most general starting position for such a linkage can me found from this configuration by a rigid change of coordinates.

With the starting position illustrated the possible rigid displacements of the end-effector relative to the base can be parameterised as,

$$g = (\lambda_1 + \mu_1 \varepsilon(\sin\phi j + \cos\phi k))(\alpha_2 + \beta_2 i)(\alpha_3 + \beta_3 j)(\lambda_4 + \mu_4 \varepsilon(\sin\psi i + \cos\psi k))$$

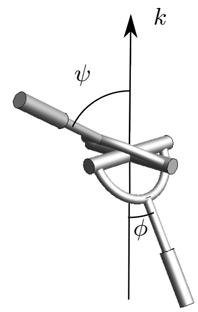

Fig. 2. General PUP linkage of the first type.

multiplying this out gives the same results as above for $a_0, \dots, a_3$ but for the c_i coordinates the results are,

$$\begin{aligned}
c_0 &= -(\lambda_1\mu_4\cos\psi + \mu_1\lambda_4\cos\phi)\beta_2\beta_3 - (\mu_1\lambda_4\sin\phi)\alpha_2\beta_3 - (\lambda_1\mu_4\sin\psi)\beta_2\alpha_3, \\
c_1 &= (\lambda_1\mu_4\cos\psi - \mu_1\lambda_4\cos\phi)\alpha_2\beta_3 + (\mu_1\lambda_4\sin\phi)\beta_2\beta_3 + (\lambda_1\mu_4\sin\psi)\alpha_2\alpha_3, \\
c_2 &= -(\lambda_1\mu_4\cos\psi - \mu_1\lambda_4\cos\phi)\beta_2\alpha_3 + (\mu_1\lambda_4\sin\phi)\alpha_2\alpha_3 + (\lambda_1\mu_4\sin\psi)\beta_2\beta_3, \\
c_3 &= (\lambda_1\mu_4\cos\psi + \mu_1\lambda_4\cos\phi)\alpha_2\alpha_3 - (\mu_1\lambda_4\sin\phi)\beta_2\alpha_3 - (\lambda_1\mu_4\sin\psi)\alpha_2\beta_3.
\end{aligned}$$

Since the results for the a_is are the same, the variety will still satisfy, $a_0a_3 - a_1a_2 = 0$. Also since these are just group multiplications the equation for the Study quadric will also be satisfied. The parametrisation also satisfies the quadratic equation,

$$\begin{aligned}
&(-2a_0\cos\phi\cos\psi + a_1\cos\psi\sin\phi + a_2\cos\phi\sin\psi - a_3\sin\phi\sin\psi)c_0 + \\
&\qquad (a_0\cos\psi\sin\phi - a_2\sin\phi\sin\psi - a_3\cos\phi\sin\psi)c_1 + \\
&\qquad (a_0\cos\phi\sin\psi - a_1\sin\phi\sin\psi - a_3\cos\psi\sin\phi)c_2 - \\
&\quad (a_0\sin\phi\sin\psi + a_1\cos\phi\sin\psi + a_2\cos\psi\sin\phi + 2a_3\cos\phi\cos\psi)c_3 = 0.
\end{aligned}$$

This quadratic equation, and the other quadratic equations in the Study coordinates, can each be written as an 8×8 symmetric matrix. Partitioned into 4×4 sub-matrices the quadratic has the shape,

$$\mathbf{g}^T\mathcal{Q}\mathbf{g} = (\bar{a}^T,\, \bar{c}^T)\begin{pmatrix} 0 & \bar{M} \\ \bar{M} & 0 \end{pmatrix}\begin{pmatrix} \bar{a} \\ \bar{c} \end{pmatrix},$$

where $\mathbf{g}$ is the vector of Study coordinates $\mathbf{g} = (a_0,\, a_1,\, a_2,\, a_3,\, c_0,\, c_1,\, c_2,\, c_3)^T$ which can be written in partitioned form with $\bar{a} = (a_0,\, a_1,\, a_2,\, a_3)^T$ and $\bar{c} =$

$(c_0, c_1, c_2, c_3)^T$. The symmetric matrix $\bar{M}$ is given by,

$$\bar{M} = \begin{pmatrix} -2\cos\phi\cos\psi & \cos\psi\sin\phi & \cos\phi\sin\psi & -\sin\phi\sin\psi \\ \cos\psi\sin\phi & 0 & -\sin\phi\sin\psi & -\cos\phi\sin\psi \\ \cos\phi\sin\psi & -\sin\phi\sin\psi & 0 & -\cos\psi\sin\phi \\ -\sin\phi\sin\psi & -\cos\phi\sin\psi & -\cos\psi\sin\phi & -2\cos\phi\cos\psi \end{pmatrix}.$$

Now suppose the PUP leg is subject to a rigid displacement. What will happen to the three quadratic equations that specify the displacement sub-variety of the linkage? Under such a displacement the group element of Study coordinates transforms according to,

$$\begin{pmatrix} a_0 \\ \mathbf{a} \\ c_0 \\ \mathbf{c} \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & TR & 0 & R \end{pmatrix} \begin{pmatrix} a_0 \\ \mathbf{a} \\ c_0 \\ \mathbf{c} \end{pmatrix}.$$

Here $\mathbf{a} = (a_1, a_2, a_3)^T$, $\mathbf{c} = (c_1, c_2, c_3)^T$, R is the 3×3 rotation matrix of the displacement and T is the $\times 3$ antisymmetric matrix corresponding to the translation vector. This means that the symmetric matrix of the quadric $\mathcal{Q}$ transforms under such a displacement to,

$$\mathcal{Q} \longmapsto \begin{pmatrix} \bar{R} & 0 \\ \bar{B} & \bar{R} \end{pmatrix}^{-T} \begin{pmatrix} 0 & \bar{M} \\ \bar{M} & 0 \end{pmatrix} \begin{pmatrix} \bar{R} & 0 \\ \bar{B} & \bar{R} \end{pmatrix}^{-1} = \begin{pmatrix} \bar{W} & \bar{U} \\ \bar{U} & 0 \end{pmatrix}$$

where

$$\bar{R} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \qquad \text{and} \qquad \bar{B} = \begin{pmatrix} 0 & 0 \\ 0 & TR \end{pmatrix}$$

and thus,

$$\bar{W} = \bar{R}(\bar{B}\bar{M} + \bar{M}\bar{B}^T)\bar{R}^T \qquad \text{and} \qquad \bar{U} = \bar{R}\bar{M}\bar{R}^T.$$

This final form is really all that will be used in the following.

Returning to the first quadric the displacement variety lies on; $a_0a_3 - a_1a_2 = 0$, this can also be written as a partitioned 8×8 matrix,

$$\mathcal{K} = \begin{pmatrix} \bar{N} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{where} \qquad \bar{N} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence after a rigid displacement this will be transformed to,

$$\mathcal{K} \longmapsto \begin{pmatrix} \bar{R}\bar{N}\bar{R}^T & 0 \\ 0 & 0 \end{pmatrix}.$$

Now consider a parallel structure consisting of base and platform connected by 3 PUP legs of this type. The possible assembly configurations of such a structure must lie on the intersection of seven quadrics in $\mathbb{P}^7$; the Study quadric and three quadrics of each of the forms,

$$(\bar{a}^T, \bar{c}^T)\mathcal{Q}_i \begin{pmatrix} \bar{a} \\ \bar{c} \end{pmatrix} = 0 \quad \text{and} \quad (\bar{a}^T, \bar{c}^T)\mathcal{K}_i \begin{pmatrix} \bar{a} \\ \bar{c} \end{pmatrix} = 0, \qquad i = 1, 2, 3.$$

These seven quadrics all contain the 3-plane of ideal points A_∞, so the intersection is not a complete intersection. However, only the residual variety, the one remaining after A_∞ has been deleted, is of interest since the points on the ideal 3-plane do not correspond to physical assemblies of the mechanism.

So consider a linear projection with centre A_∞ onto the 3-plane given by $c_0 = c_1 = c_2 = c_3 = 0$. This second plane will be called A_0, the image of the map is usually known as the spherical indicatrix of the mechanism. Using the 4 quadrics, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and the Study quadric $\mathcal{Q}_s$ the projection can be inverted. Notice that the 4 quadrics can be thought of as linear equations in the c_i coordinates, that is,

$$2\bar{a}^T\bar{c} = 0 \quad \text{and} \quad 2\bar{a}^T\bar{U}_i\bar{c} = -\bar{a}^T\bar{W}_i\bar{a}, \qquad i = 1,\, 2,\, 3,$$

where the first equation is the Study quadric. Let $\tilde{a}$ be a point in A_0 and form the 4×4 matrix $\bar{\Lambda}$ which has rows $2\tilde{a}^T\bar{U}_1$, $2\tilde{a}^T\bar{U}_2$, $2\tilde{a}^T\bar{U}_3$ and $2\tilde{a}^T$. Also write $\bar{\eta}$ as the 4-vector with entries, $-\tilde{a}^T\bar{W}_1\tilde{a}$, $-\tilde{a}^T\bar{W}_2\tilde{a}$, $-\tilde{a}^T\bar{W}_3\tilde{a}$ and 0. The 4 quadrics can be written in the matrix form,

$$\bar{\Lambda}\bar{c} = \bar{\eta}.$$

Using linear algebra we can invert the the equation to produce a point in $\mathbb{P}^7$,

$$\bar{a} = \det(\bar{\Lambda})\tilde{a} \qquad \text{and} \qquad \bar{c} = \mathrm{Adj}(\bar{\Lambda})\bar{\eta}$$

where Adj denotes the transposed matrix of cofactors usually called the adjugate matrix. So given a point $\tilde{a}$, in A_0 this gives a rational map to the intersection of the 4 quadrics in $\mathbb{P}^7$. The map clearly inverts the projection to A_0 and hence shows that the intersection of the 4 quadrics (with the component A_∞ deleted) is birationally equivalent to a $\mathbb{P}^3$, namely A_0. Notice that the image of a point $\tilde{a}$ has degree 5 in the coordinates of $\tilde{a}$.

To project the other three quadrics $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ to A_0 the coordinates c_0, c_1, c_2 and c_3 must be eliminated, but the quadrics do not contain these variables. Hence the quadrics project to three 2-dimensional quadrics in A_0. In general the intersection of three such quadrics will give $2^3 = 8$ discrete points and, after mapping back to $\mathbb{P}^7$, this gives 8 possible assembly modes for the structure.

There are however, a number of possible special cases. These are cases where the displacement sub-variety of the 3 PUP parallel mechanism has a component of one or two dimensions and hence the machine acquires one or two degrees of freedom. Two particular cases are particularly simple. Suppose the U joints of the three legs are all parallel. That is, in the home position, the U joints are simply translates of each other with no rotation. In this case the three quadrics

$\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are identical and hence the mechanism will be 2-dimensional. The displacement sub-variety will be birationally equivalent to a 2-dimensional quadric in $\mathbb{P}^3$. In the second special case, suppose that only two of the U joints are parallel in the home position. Now the spherical indicatrix is the intersection of a pair of 2-dimensional quadric and so is, in general, an elliptic quartic curve.

It is also possible that the three 2-dimensional quadrics intersect in a curve, necessarily not a complete intersection. Only a few possibilities can arise, for the intersection:

- A twisted cubic curve.
- A conic and two discrete points.
- Two lines and a two discrete points.
- A single line and four discrete points.

See for example, [7].

The question arises, which of these possibilities can occur as the displacement variety of a 3 UPU mechanism? Remember that each of the 3 quadrics are related to each other by rigid rotations. One possibility that can occur is the line and 4 points. This is because the three legs can be arranged in such a way that the U joints in two of the legs are rotated around a line in the quadric determined by the third leg. This line will thus lie in all three quadrics and hence their intersection will consist of the line and four discrete points, some or all of these discrete points might be complex points however. Note that since the rotational motion of the platform will be a rotation about a fixed direction here, the complete motion of the platform will be a Schönflies motion. Moreover, it can be seen that the image of a line in A_0 will generally be a curve in the Study quadric with degree 5.

4 Second and Third Types

The second and third types of PUP linkage as shown in fig. 1, are distinguished by the fact that direction of one of the P joints is parallel to the axis of its neighbouring R joint. For the second type the two joints with parallel axes are the last two while in the third type they are the first two. This type of linkage could be analysed in a similar fashion to the work in the previous section. But in these cases there is a simpler method because here the displacement sub-variety can be identified with a purely geometric problem. Consider for definiteness the second type of linkage here. Call the axis of the final R joint ℓ, the direction of this line is parallel to the direction of the final P joint. Also let π be the plane through the centre of the U joint and perpendicular to the first R joint of the universal joint. Now it is clear that the line ℓ lies in the plane π and any motion of the linkage will preserve this incidence.

The variety of displacements which move a line so that it remains in a fixed plane can be see to be a Segre variety $\mathbb{P}^3 \times \mathbb{P}^1$. This is a product of subgroups: a rotation about the line ℓ, in its initial configuration can be followed by a planar displacement taking π to itself. According to [2], the degree of a Segre variety $\mathbb{P}^3 \times \mathbb{P}^1$ is $\binom{3+1}{1} = 4$. The space of displacements that this second type of PUP can achieve will be an open set in this Segre variety. This is because the linkage cannot move the axis of the final R joint parallel to the axis of the first R joint. These impossible displacements comprise translations in the plane but parallel to the first R joint combined with any rotation about the second R joint. These impossible displacements form a $\mathbb{P}^1 \times \mathbb{P}^1$ Segre variety and so the displacements that the second type of PUP linkage can achieve are a $\mathbb{P}^3 \times \mathbb{P}^1$ with a $\mathbb{P}^1 \times \mathbb{P}^1$ deleted.

Clearly the third type of PUP linkage is the inversion of the second type so the methods of [8] could be used to study its displacement variety. However, it is clear that the possible displacements of the end-effector can be viewed as the displacements of a plane such that the plane always contains a fixed line. This is simply a planar motion followed by a rotation about the line and hence is also be a Segre variety $\mathbb{P}^1 \times \mathbb{P}^3$. Hence the displacements that this third type of PUP linkage can achieve also comprise an open set in a $\mathbb{P}^1 \times \mathbb{P}^3$ Segre variety.

It is well know, see [2] for example, that a $\mathbb{P}^3 \times \mathbb{P}^1$ Segre variety lies in 6 linearly independent quadrics in $\mathbb{P}^7$. For the second type of PUP with home position illustrated in fig. 1 above the 6 quadrics can be found as follows. The dual quaternions representing the rigid displacement of the end-effector can be written,

$$g = (\lambda_1 + \mu_1 \varepsilon k)(\alpha_2 + \beta_2 i)(\alpha_3 + \beta_3 k)(\lambda_4 + \mu_4 \varepsilon k),$$

where, as above, the α_i, λ_j and so forth are the homogeneous parameters of the joints. Expanding this equation and collecting terms gives,

$$\begin{aligned} a_0 &= \lambda_1\lambda_4\alpha_2\alpha_3, & c_0 &= -(\lambda_1\mu_4 + \mu_1\lambda_4)\alpha_2\beta_3, \\ a_1 &= \lambda_1\lambda_4\beta_2\alpha_3, & c_1 &= (\lambda_1\mu_4 - \mu_1\lambda_4)\beta_2\beta_3, \\ a_2 &= -\lambda_1\lambda_4\beta_2\beta_3, & c_2 &= (\lambda_1\mu_4 - \mu_1\lambda_4)\beta_2\alpha_3, \\ a_3 &= \lambda_1\lambda_4\alpha_2\beta_3, & c_3 &= (\lambda_1\mu_4 + \mu_1\lambda_4)\alpha_2\alpha_3. \end{aligned}$$

The six quadratic equations satisfied by these group elements can be expressed as,

$$\text{Rank}\begin{pmatrix} a_0 & a_1 & c_2 & c_3 \\ a_3 & -a_2 & c_1 & -c_0 \end{pmatrix} = 1.$$

That is the six possible 2×2 sub-determinants of this matrix must vanish.

To find the quadrics for the general case the geometric problem of displacing a general line ℓ so that it remains on a fixed plane π could be considered. There are several other ways to express this geometry, suppose ℓ_1, ℓ_2 and ℓ_3 are three linearly independent lines lying in the plane π, then any rigid displacement which keeps ℓ in the plane π will also preserve the reciprocity between ℓ and the three other lines ℓ_1, ℓ_2, ℓ_3. This will only produce 3 quadrics however.

5 Conclusion

The results here suggest the novel possibility that a mobile over-constrained parallel manipulator can have assembly configurations of different dimensions. This phenomenon does not seem to have attracted much attention in the parallel robotics literature to date. It is however, well known for closed loop mechanisms such as the cyclohexane molecule.

Associating a linkage with a purely geometrical problem provides a powerful technique for understanding the linkage's displacement variety. There are now many examples in the literature, probably beginning with the identification of the spherical-spherical linkage with the problem of moving a point in such a manner that it remains on a fixed sphere. The first PUP linkage considered above can also be associated with a geometrical problem. Consider a pair of lined planes. A lined plane in a plane with a distinguished line lying in the plane. In terms of the linkage the two planes are the planes normal to the two R joints and passing through the point where the axes of the R joints cross. The distinguished lines are formed by translating the centre point of the U joint in the direction of the first or last P joints. Now assume that the first lined plane is fixed and consider the possible rigid displacements of the second which move it in such a way that the planes remain perpendicular and the lines remain in contact. It is not too difficult to see that these are precisely the possible displacements of the end-effector of this PUP linkage relative to its home position.

Limitations of space has precluded the study of a fourth type of PUP linkage where the direction of the final P joint lies in the same plane as the axes of the two R joints. Preliminary results suggest that the displacement variety of such a linkage is not too different from the that of the first type considered here.

References

1. D. Gan and J.S. Dai, 2013, "Geometry Constraint and Branch Motion Evolution of 3-PUP Parallel Mechanisms with Bifurcated Motion", *Mechanism and Machine Theory*, **61**:168-183. http://dx.doi.org/10.1016/j.mechmachtheory.2012.09.011.
2. J. Harris, 1992, *Algebraic Geometry a first course*, Springer Verlag, New York.
3. F. Polizzi, 2014, "Degree of the Segre product of lines", (https://mathoverflow.net/users/7460/francesco-polizzi), URL (version: 2014-06-11).
4. E. Rodriguez-Leal, J.S. Dai and G.R. Pennock, 2009, "Inverse kinematics and motion simulation of a 2-DOF parallel manipulator with 3-PUP legs" in *Computational Kinematics, Proceedings of the 5th International Workshop on Computational Kinematics*, Editors: A. Kecskeméthy, A., Müller, Springer Verlag Berlin Heidelberg, pp.85–92. https://doi.org/10.1007/978-3-642-01947-0_11.
5. J.M. Selig, 2014, "Some remarks on the *RRR* linkage", in *Advances in Robot Kinematics*, Editors, J. Lenarčič and O. Khatib, Springer International Publishing, pp. 77–85. doi:10.1007/978-3-319-06698-1_9
6. J.M. Selig, 2005, *Geometric Fundamentals of Robotics*, Springer Verlag, New York.
7. J.G. Semple and L. Roth, 1949, *Introduction to Algebraic Geometry.* Clarendon Press, Oxford. Reprinted in 1985 with index.
8. T. Stigger, M.L. Husty, 2018, "Constraint Equations of Inverted Kinematic Chains". In: Zeghloul S., Romdhane L., Laribi M. (eds) *Computational Kinematics. Mechanisms and Machine Science*, vol 50. Springer, Cham. https://doi.org/10.1007/978-3-319-60867-9_56
9. K. Zhang, J. S. Dai and Y, Fang, 2012,"Constraint analysis and bifurcated motion of the 3PUP parallel mechanism", *Mechanism and Machine Theory* **49**:256-269. doi:10.1016/j.mechmachtheory.2011.10.004.