Brazier deformation of elliptical section tubes

F. McCann, A. Wadee, L. Gardner

Abstract

When tubular members are in bending, they tend to flatten towards the axis of bending, thus reducing their second moment of area. Eventually, a limit point is reached whereupon the initial stability of the system is lost and unstable equilibrium prevails, thus reducing the load-carrying capacity of the member. This effect was originally described by Brazier (1927) for circular tubular members. In the present study, the analysis of Brazier is adapted for elliptical hollow section members, taking into account the additional geometric complexities inherent in ellipses. An analytical method is presented whereby the initial geometry and the displacement functions of the system are replaced by Fourier series, thus reducing the analytical complexity of the problem. After formulating the potential energy functional, use of a variational method allows for the amplitudes of the constituent harmonics of the Fourier approximations of the displacement functions to be solved for, providing estimates of the deformed geometry of the cross-section and the associated moment. In keeping with the analogy of Brazier for circular sections, a limit point is observed. These analytical predictions are then compared with the results of a complementary finite element analysis, whereupon it is found that for smaller longitudinal curvatures there is close agreement between the analytical and numerical methods. For larger curvatures and moments beyond the limit point some divergence is observed between the predictions of the two methods, which can be attributed to the lower-order approximations assumed in formulating the potential energy functional in the analytical method

Symbols used

1 Introduction

In recent years, steel elliptical hollow section (EHS) members have become of more interest due to their introduction and availability as hot-finished products [1], their aesthetic properties and their enhanced flexural properties compared to circular hollow section (CHS) tubes [2]. Research on hollow steel tubes of elliptical cross-section has been extensive in recent years, including the testing and complementary numerical analysis of such members in concentric and eccentric compression [3, 4] and in bending [5]. The buckling of steel EHS columns and beams was investigated by [6] and [2, 7], respectively, while local postbuckling behaviour was examined by [8]. An analytical study of the buckling of EHS tubes under compression was conducted by [9]. These studies provided a basis upon which design rules for steel EHS members could be formulated [10], including cross-section classification [11], compressive resistance [3], bending [5] and flexural buckling [6].

In the current work, a structural EHS member is subjected to pure bending about either its major or minor axis. As described in the seminal work of Brazier [12], the cross-section of a circular tube in bending tends to flatten towards the axis of bending and hence some flexural rigidity is lost since the second moment of area is reduced. Owing to the increasing loss of rigidity, and hence moment-carrying capacity, with increasing curvature, a limit point in the moment–curvature relationship for the system is reached, as shown in Figure 1. At curvatures greater than this limit point, the system is now in unstable equilibrium and the moment resistance of the tubular member is compromised.

The present study examines the Brazier effect for EHS sections. From an analytical perspective, difficulties arise due to the varying radius of curvature of the wall of an elliptical section; such difficulties are not encountered during an equivalent analysis of a circular tube. To overcome this, Fourier series are used to model the original cross-sectional geometry of the elliptical tubes. The problem of determining the deflected shape of the cross-section is solved using a Rayleigh–Ritz approach whereby the displacement functions, namely the radial and circumferential components of the deflection of the tube wall, are represented by Fourier series, as has been performed previously for I-section beams restrained at discrete points [13]. The system is converted into one where the amplitudes of the harmonics constituting the displacement functions become the generalised coordinates of the system and can be solved for after imposing equilibrium conditions.

With the solution for the displacement functions found, an overall moment–curvature relationship for the EHS member can be found, along with the limit point mentioned previously, thus allowing the moment capacity of the tube in this bending mode to be determined. This analytical method is then validated against a numerical model developed with the finite element software ABAQUS [14], with very close agreement observed between the analytical and numerical results for low curvatures with the results diverging as the curvature increases.

2 Analytical method

In this section, an analytical approach to determining the behaviour of an EHS beam in bending is described. After outlining the geometry of the problem and developing the potential energy functional of the system, a Rayleigh–Ritz approach is applied whereby the displacement functions are represented by Fourier series. The amplitudes of the constituent harmonics of the displacement functions become the generalized coordinates of the system and the equilibrium of the system is analysed by taking the partial derivatives of the potential energy functional with respect to these coordinates. The results of this analysis are compared to the results of a complementary numerical model in Section 3.

2.1 Geometry

In the present study, the stability of an elliptical tube of length L of cross-sectional radii a and b in bending is examined. It is assumed that the tube is under a uniform bending moment M imposing a constant curvature χ throughout the tube. In order to facilitate the analysis of tubes being bent about either the major axis or minor axis, the cross-sectional axial dimensions a and b are defined as those parallel to the global x and y axes, respectively, as shown in Figure 2, so that the tube is always bent about the x axis. The global z axis runs longitudinally along the span of the tube. The value of the aspect ratio b/a indicates about which axis the tube is being bent – for $b/a > 1$, the tube is being bent about its major axis, while for $b/a < 1$, the tube is being bent about its minor axis. The thickness t of the tube wall is assumed to be small compared to the radii and so hereafter points on the midsurface of the tube wall are considered representative of the full thickness of the tube wall. The origin of the Carteisan coordinate system is located at the centre of the ellipse, and so the x and y ordinates of a point on the midsurface of the elliptical section can be expressed in terms of a parametric angle α as:

$$x = a\cos\alpha, \qquad (1)$$

$$y = b\sin\alpha.$$

The radius of curvature r, as opposed to the distance from the centre of the cross-section, is used in the present study since it facilitates a more direct and convenient determination of the arclength of a curve. For an elliptical section, the radius of curvature is given by the following expression [9]:

$$r(\alpha) = \frac{(a^2\cos^2\alpha + b^2\sin^2\alpha)^{\frac{3}{2}}}{ab}. \tag{2}$$

The angle θ made by the radius of curvature with the global x axis is related to α by the expression:

$$\tan\theta = \frac{b}{a}\tan\alpha \tag{3}$$

In the present study, integration around the circumference of the elliptical section is performed in terms of θ for convenience. Although an analytical form of r in terms of θ can be found, it does not lend itself easily to integration and thus it is more convenient to express r and its reciprocal, the local tube wall curvature κ, as Fourier series thus:

$$r(\theta) = \sum_{i=0}^{N_G} r_{2i}\cos 2i\theta, \tag{4}$$

$$\kappa(\theta) = \sum_{i=0}^{N_G} \kappa_{2i}\cos 2i\theta \tag{5}$$

where N_G is the number of terms used in order to provide an acceptable convergence with the analytical representations of the geometrical functions. The values of the coefficients r_n and κ_n are determined using least square regression against the value of the analytical functions in order to provide the most accurate fit.

Since the analysis is carried out in terms of θ rather than α, it is necessary to represent the function $\sin\alpha$ in terms of θ. This is again performed using a Fourier series:

$$\sin\alpha = \sum_{i=1}^{N_G} y_{2i-1}\sin(2i-1)\pi\theta \tag{6}$$

In the case of $b/a = 1.0$, i.e., a circular tube, the Fourier approximations for r and κ collapse to the constant values a and $1/a$, respectively while $\sin\alpha$ collapses to $\sin\theta$.

2.2 Displacement field

At an arbitrary point on the midsurface of the wall of the elliptical section, two displacement functions are defined: the radial displacement u and the circumferential displacement v, which are oriented to form a right-handed orthogonal local coordinate system together with the longitudinal axis z, as shown in Figure 2. The displacements u and v both contain a linear portion, u_o and v_o, respectively, related to the St. Venant solution and a nonlinear portion, u_n and v_n, respectively. The St. Venant solutions are given by:

$$u_{\mathrm{o}} = -\nu\chi y(\theta)r(\theta) \tag{7}$$

$$v_{\mathrm{o}} = -\int u_{\mathrm{o}} \mathrm{d}\theta \tag{7}$$

The two nonlinear displacements are assumed to form an inextensional system. An additional radial displacement u_{n} of the circumferential element is accompanied by a change in the circumferential displacement $\mathrm{d}v_{\mathrm{n}}$. After including the St. Venant displacements, an element of deformed arclength $\mathrm{d}s_{\mathrm{o}}$ and radius of curvature r_{o} is considered. The element is further deformed by the inextensional system of displacements so that the radius of curvature is increased by u_{n} and the arclength is increased by $\mathrm{d}v_{\mathrm{n}}$ to $\mathrm{d}s_{\mathrm{n}}$:

$$\mathrm{d}s_{\mathrm{n}} = \mathrm{d}s_{\mathrm{o}} + \mathrm{d}v_{\mathrm{n}} = (r + u_{\mathrm{n}})\mathrm{d}\theta \tag{8}$$

$$\Rightarrow u_{\mathrm{n}} = \frac{\mathrm{d}v_n}{\mathrm{d}\theta}.$$

2.3 Energy formulation

Since the bending moment and longitudinal curvature are considered constant along the length of the member, it suffices to examine a particular cross-section of the beam. The strain energy at a given cross-section along the EHS member is assumed to have two components, related to the longitudinal strain and the change in local curvature of the tube wall, respectively. Assuming Euler-Bernoulli bending, the longitudinal strain ε_z is proportional to the perpendicular distance between the point on the midsurface of the tube wall and the axis of bending, which, when including the contributions of the radial and circumferential displacements as shown in Figure 2, is given by:

$$\varepsilon_z = \chi(b \sin\alpha + u \sin\theta - v \cos\theta). \tag{9}$$

At a given cross-section, the strain energy per unit length of the tube is given by:

$$U = \frac{1}{2}\int \sigma\varepsilon \, dA. \tag{10}$$

The initial area of an infinitesimal element of the tube wall is $\mathrm{d}A = t\mathrm{d}s = tr\mathrm{d}\theta$, which after deformation become $t(r+u)\mathrm{d}\theta$. The material is assumed to be linearly elastic with a modulus of elasticity E, therefore:

$$U = \frac{1}{2}E\int_0^{2\pi} \varepsilon^2 t(r+u)\mathrm{d}\theta. \tag{11}$$

Thus, the strain energy arising from longitudinal curvature is:

$$U_{\text{long}} = \frac{1}{2}E\int_0^{2\pi} \chi^2(b\sin\alpha + u\sin\theta - v\cos\theta)^2 t(r+u)\mathrm{d}\theta. \quad (12)$$

The change in curvature of the tube wall is derived by considering two points A and B on the deformed surface of the tube wall with initial radii of curvature r and r + dr, respectively, that have displaced radially by distances u and u + du, respectively, as shown in Figure 3. The local curvatures κ at the two points are initially 1/r and 1/(r + dr), respectively, and the arclength between the two points is initially ds = rdθ. After deformation the arclength has increased to ds' = (r + u)dθ. Adapting the analysis of thin-walled circular tubes in bending by Den Hartog (1952) to elliptical sections, the slope ϕ of the deformed surface at point A relative to the dashed line in Figure 3 is given by:

$$\phi_{\text{A}} = \frac{\mathrm{d}u}{\mathrm{d}s'} = \frac{\mathrm{d}u}{(r+u)\mathrm{d}\theta} = \frac{u'}{r+u}, \quad (13)$$

where dashes denote differentiation with respect to θ. Expanding then as a Taylor series:

$$\phi_{\text{A}} = \frac{u'}{r+u} = \frac{u'}{r}\left(1 - \frac{u}{r} + \cdots\right). \quad (14)$$

For the purposes of the present study, deformations are assumed small and the series is truncated after the first term, so that:

$$\phi_{\text{A}} = \frac{u'}{r}. \quad (15)$$

The slope at B ϕ_{B} relative to the dashed line in Figure 3 is given by:

$$\phi_{\text{B}} = \phi_{\text{A}} + \frac{\mathrm{d}}{\mathrm{d}\theta}(\phi_{\text{A}})\mathrm{d}\theta = \frac{u'}{r} + \left(\frac{u''}{r} - \frac{u'r'}{r^2}\right)\mathrm{d}\theta. \quad (16)$$

The difference in slope between A and B with respect to a fixed direction is:

$$\mathrm{d}\phi = \mathrm{d}\theta - \phi_{\text{B}} + \phi_{\text{A}} = \left(1 - \frac{u''}{r} + \frac{u'r'}{r^2}\right)\mathrm{d}\theta. \quad (17)$$

Therefore, the local curvature of the deformed tube wall is:

$$\kappa_{\text{L}} = \frac{\mathrm{d}\phi}{\mathrm{d}s'} = \frac{\mathrm{d}\phi}{(r+u)\mathrm{d}\theta} = \frac{\left(1 - \frac{u''}{r} + \frac{u'r'}{r^2}\right)}{r+u}. \quad (18)$$

Expanding again as a Taylor series:

$$\kappa_{\text{L}} = \frac{1}{r}\left(1 - \frac{u''}{r} + \frac{u'r'}{r^2}\right)\left(1 - \frac{u}{r} + \cdots\right). \quad (19)$$

Noting that the initial curvature was 1/*r*, and neglecting higher-order terms, the change in local curvature is thus given by:

$$d\kappa_L = \frac{u}{r^2} + \frac{u''}{r^2} - \frac{u'r'}{r^3}. \tag{20}$$

In the case of a circular tube, the third term in this approximation vanishes owing to *r* being constant in θ. This change in local curvature in the tube wall is essentially analogous to a plate in bending, and thus the strain energy is given by:

$$U_{\text{local}} = \frac{1}{2}E\int_0^{2\pi} \frac{t^2}{12(1-\nu^2)}\left(\frac{u}{r^2} + \frac{u''}{r^2} - \frac{u'r'}{r^3}\right)^2 tr\mathrm{d}\theta, \tag{21}$$

where ν is the Poisson's ratio. Summing the two strain energy contributions results in the total strain energy:

$$U = \frac{1}{2}E\int_0^{2\pi} \left(\chi^2(b\sin\alpha + u\sin\theta - v\cos\theta)^2 + \frac{t^2}{12(1-\nu^2)}\left(\frac{u}{r^2} + \frac{u''}{r^2} - \frac{u'r'}{r^3}\right)^2\right) tr\mathrm{d}\theta. \tag{22}$$

Following from Eq.8, the expression for *U* can be written in terms of the circumferential displacement function *v* as follows:

$$U = \frac{1}{2}E\int_0^{2\pi} \left(t(r+u)\chi^2(b\sin\alpha + v'\sin\theta - v\cos\theta)^2 + \frac{t^3 r}{12(1-\nu^2)}\left(\frac{v'}{r^2} + \frac{v'''}{r^2} - \frac{v''r'}{r^3}\right)^2\right) \mathrm{d}\theta. \tag{23}$$

Considering that the longitudinal curvature χ is the first derivative of the slope of the beam with respect to the longitudinal coordinate *z*, the work done per unit length of the tube by the constant bending moment *M* is given by:

$$W = M\chi. \tag{24}$$

Thus, the potential energy functional of the system at a given cross-section is:

$$V = \frac{1}{2}E\int_0^{2\pi} \left(t(r+u)\chi^2(b\sin\alpha + v'\sin\theta - v\cos\theta)^2 + \frac{t^3 r}{12(1-\nu^2)}\left(\frac{v'}{r^2} + \frac{v'''}{r^2} - \frac{v''r'}{r^3}\right)^2\right) \mathrm{d}\theta - M\chi. \tag{25}$$

2.4 Governing differential equation

At this juncture, the calculus of variations can be applied in order to determine the governing differential equations the system. The potential energy functional is recast in terms of a Lagrangian:

$$V = \int_0^{2\pi} \mathcal{L}(v, v', v'', v''')\mathrm{d}\theta. \tag{26}$$

For equilibrium conditions to prevail, the first variation of the total potential energy δV, must vanish. The first variation is given by:

$$\delta V = \int_0^{2\pi} \left(\frac{\partial \mathcal{L}}{\partial v}\delta v + \frac{\partial \mathcal{L}}{\partial v'}\delta v' + \frac{\partial \mathcal{L}}{\partial v''}\delta v'' + \frac{\partial \mathcal{L}}{\partial v'''}\delta v'''\right)\mathrm{d}\theta. \tag{27}$$

Due to the complexity of the differential equation, numerical continuation methods are required to solve the differential equation for v, which runs contrary to the motive of the present study to develop an analytical approach to solving the problem.

2.5 Rayleigh–Ritz approach

Instead of solving the governing differential equations for the displacement functions directly, the Rayleigh–Ritz approach of substituting approximate functions for the displacement functions can be performed. In the present study, a Fourier series is substituted for the circumferential displacement function v; this is particularly appropriate considering the inherently periodic nature of the cross-sectional deformations of a closed section. The displacement v can be represented by the following series:

$$v = \sum_{i=1}^{N_\mathrm{F}} v_{2i} \sin 2i\theta, \tag{28}$$

where v_n are the amplitudes of the constituent harmonics of the Fourier series, and N_F is the number of Fourier modes utilised in approximating the displacement function. It was found that setting $N_\mathrm{F} = 5$ provided adequate convergence of the Fourier series. It is noted that, due to the doubly-symmetric geometry and boundary conditions, only even harmonics are required; this has been confirmed by comparison with analysis including all harmonics. When substituted into the potential energy functional, the amplitudes v_n now become the generalised coordinates of the system. The deformed shape of the elliptical cross-section is determined through simultaneous solution of the set of equilibrium equations:

$$\frac{\partial V}{\partial v_n} = 0, \tag{29}$$

for the amplitudes v_n. Since the work done, $M\chi$, is constant at every cross-section, it does not contribute terms to the equilibrium equations, and thus the equilibrium of the system can be assessed by examination of the strain energy alone.

2.6 Results of Rayleigh–Ritz analysis

The mathematical computation software Maple [15] was used to carry out the integration of the strain energy and the subsequent solution of the equilibrium equations for v_i. Four cross-section geometries were analysed: b/a = 2, b/a = 1.5 (both representing bending about the major axis), b/a = 0.5 and b/a = 0.666 (representing bending about the minor axis). Since commercially-available EHS tubes have an aspect ratio of 2.0, the b/a = 2 and b/a = 0.5 cases are of most practical interest, while the b/a = 0.666 and b/a = 1.5 cases are intended to demonstrate the generality of the method. To facilitate computation, occurrences of $1/r$ in the potential energy functional are replaced by κ. The values of the amplitudes of the constituent harmonics of the Fourier series approximations of r, κ and sin α determined for each section through least square regression are shown in Table 1. The values of r_n and κ_n are normalised by a and $1/a$, respectively, demonstrating the generality of the coefficients in Table 1 to all sections with these aspect ratios. The rapid convergence between the Fourier series approximations and the exact analytical functions is demonstrated by the fact that the maximum error for any of the functions across the three cases examined was in the order of 0.002% for six harmonics included i.e. N_G = 5. It was shown by [9] that more harmonics are required for higher aspect ratios in order to achieve similar levels of agreement between the exact functions and the Fourier series. The geometrical and material properties of the EHS tubes analysed are summarised in Table 2.

The results of the analysis for the four cases are shown in Figures 4 to 7, showing both the predicted deformed cross-sectional geometries for a range of values of longitudinal curvature χ and the moment–curvature relationships up to the limit points. It can be seen in Figure 4 for b/a = 0.5 that at higher curvatures, inward facing lobes have begun to form at the top and bottom of the section.

The moment acting at the cross-section is determined using the linear approximation:

$$M = EI(\chi)\chi, \tag{31}$$

where $I(\chi)$ is the second moment of area of the cross-section taking into account the deformations associated with a longitudinal curvature χ, and is given by:

$$I(\chi) = \int_0^{2\pi} (b\sin\alpha + u\sin\theta - v\cos\theta)^2 t(r+u)\,\mathrm{d}\theta. \tag{32}$$

3 Numerical analysis

In this section, a numerical model developed in the finite element analysis software ABAQUS [14] that simulates an EHS tube in uniaxial bending about either the major or minor axis is described. The results obtained from the numerical model are compared with those obtained from the analytical model described in Section 2.

3.1 Description of model

Three separate models were created, each corresponding to the three b/a ratios analysed in Section 2, respectively. The material properties and dimensions of the models used were identical to those shown in Table 2 for the analytical method, with linear elasticity assumed for the steel material model. A simply-supported EHS tube with a span of 10 m under equal and opposite end moments was modelled. The symmetry of the cross-section geometry and the boundary conditions was exploited so that only a quarter of the tube (of length 5 m) was modelled, with symmetry conditions imposed at the midspan cross-section and longitudinally along the tube, as shown in Figure 4. At the end of the tube, a kinematic coupling was defined linking the end cross-section to a reference point located at the centre of the section. A rotation displacement boundary condition was imposed in order to simulate the end moment. The model was meshed using 4-noded reduced integration S4R shell elements with a characteristic dimension of 5 mm. The model was solved using a static general analysis which included nonlinear geometry.

3.2 Comparison of analytical and numerical results

The analytical and numerical predicted moment–curvature relationships for the four cases are compared in Figures 9 to 12. Good agreement between the two models is observed in all cases in the initial linear range and for a portion of the nonlinear range. The results diverge at larger curvatures, most likely owing to the truncation of the Taylor series expansions involved in the calculation of the strain energy due to the change in curvature of the tube wall. This divergence leads to an over-estimation of the limit moment of the system by the analytical model since it cannot capture the full effects of the nonlinear behaviour.

The cross-sectional geometry at the midspan of the tube was extracted at a number of curvatures and is compared with the equivalent deformed cross-sections predicted by the Rayleigh–Ritz analysis in Figures 13 to 16. It can be seen that at low curvatures the analytical method agrees very well with the numerical predictions. In keeping with the divergence observed in the moment–curvature graphs, for higher curvatures the deformed geometry predicted by the two methods also diverges.

Thus it can be said that for small deflections, the analytical method provides accurate predictions of the deformed geometry and the moment–curvature behaviour. For higher curvatures approaching the limit point, the nonlinear effect of the larger deflections is not fully captured by the analytical method and the predictions diverge from the geometrically-nonlinear numerical analysis.

4 Conclusions

An analytical approach to examining the Brazier effect in elliptical section tubes in bending has been presented. The complexity inherent in the analysis of ellipses has been circumvented by approximating the cross-sectional geometry using Fourier series, greatly aiding integration about the circumference of the section. The potential energy functional of the system was developed, incorporating the effects of both longitudinal strain and the change in curvature of the tube wall. A Rayleigh–Ritz method is then applied in order to solve for the deformed geometry of the section whereby the circumferential displacement function is substituted by a Fourier series. The amplitudes of the constituent harmonics are solved for, providing a solution for the deformed geometry of the section and thus allowing the moment of resistance of the section to be calculated. These results were then compared with the results of a finite element analysis conducted using Abaqus [14]. Comparison of the analytical and numerical methods showed that for lower curvatures there was good agreement between the predictions of the two methods. For higher curvatures, the predictions of the two methods diverged. Truncation of Taylor series expansions involved in the formulation of the potential energy functional and of the Fourier series approximations of the displacement functions inhibits the ability of the analytical method to capture the nonlinear behaviour present at the higher curvatures fully.

References

1. Comité Européen de Normalisation. 2006. *EN 10210-1:2006 Hot finished structural hollow sections of non-alloy and fine grain steels – Part 1: Technical delivery conditions*. British Standards Institution.
2. Ruiz-Teran AM, Gardner L. 2008. Elastic buckling of elliptical tubes. *Thin-Walled Struct.*, **46**, 1304–1318.
3. Chan TM, Gardner L. 2008. Compressive resistance of hot-rolled elliptical hollow sections. *Eng. Struct.*, **30**, 522–532.
4. Law KH, Gardner L. 2013. Buckling of elliptical hollow section members under combined compression and uniaxial bending. *J. Const. Steel Res.*, **86**, 1–16.
5. Chan TM, Gardner L. 2009. Bending strength of hot-rolled elliptical hollow sections. *J. Const. Steel Res.*, **64**, 971–986.
6. Chan TM, Gardner L. 2009. Flexural buckling of elliptical hollow section columns. *J. Struct. Eng., ASCE*, **135**(5), 546–557.
7. Law KH, Gardner L. 2012. Lateral instability of elliptical hollow section beams. *Eng. Struct,*, **37**, 152–166.

8. Silvestre N, Gardner L. 2011. Elastic post-buckling of elliptical tubes. *J. Const. Steel Res.*, **67**, 281–292.
9. Silvestre N. 2008. Buckling behavior of elliptical cylindrical shells and tubes under compression. *Intl. J. Solids & Struct.*, **45**, 4427–4447.
10. Chan TM, Gardner L, Law KH. 2010. Structural design of elliptical hollow sections: a review. *Proc. Inst. Civil Engrs Struct. Build.*, **163**(6), 391–402
11. Gardner L., Chan TM. 2007. Cross-section classification of elliptical hollow sections. *Steel and Comp. Struct.* **7**(3), 185–200.
12. Brazier LG. 1927. On the flexure of thin cylindrical shells and other "thin" sections. *Proc. Royal Soc.* **116**, 104–114.
13. McCann F, Gardner L, Wadee MA. 2013. Lateral stability of imperfect discretely braced steel beams. *J. Eng. Mech., ASCE*, **139**(10), 1341–1349.
14. Simulia Inc. 2012. *Abaqus 6.13 User's Manual*, Simulia.
15. Maplesoft Inc. 2014. *Maple 18 Manual*, Maplesoft