The Complex of Lines from Successive Points and the Horopter

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Pinhole Camera

Very simple but common model of a camera. Can put coordinates on the image plane, don't need to here, just think about the lines through the optical centre.

Plücker Coordinates

Given 2 points,

$$\mathbf{p}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \text{and} \quad \mathbf{p}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Line joining them has direction $\mathbf{p}_1 - \mathbf{p}_2$ and moment $\mathbf{p}_1 \times \mathbf{p}_2$.

Line's Plücker coordinates are these 6 quantities—written as a 6-vector,

$$\begin{pmatrix} p_{01} \\ p_{02} \\ p_{03} \\ p_{23} \\ p_{31} \\ p_{12} \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \\ y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 z_2 \end{pmatrix}.$$

Plücker Coordinates cont.

Partitioned form for Plücker coordinates,

$$\begin{pmatrix} \mathbf{p}_1 - \mathbf{p}_2 \\ \mathbf{p}_1 \times \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}.$$

Lines in $\mathbb{P}^3$, homogeneous coordinates $\tilde{\mathbf{p}}^T = (x_1, x_2, x_3, x_0)$. Now Plücker coordinates just $p_{ij} = x_i y_j - x_j y_i$ for $i, j = 0, 1, 2, 3$. Get back to $\mathbb{R}^3$ by putting $x_0 = 1$.

The Klein Quadric

Using different points on the same line gives the same Plücker coordinates multiplied by a non-zero constant. Plücker coordinates are homogeneous coordinates in $\mathbb{P}^5$.

Not all points in $\mathbb{P}^5$ represent lines, all lines must satisfy the homogeneous quadratic relation $p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$. In partitioned form, if $\mathbf{s}^T = (\boldsymbol{\omega}^T, \mathbf{v}^T)$, then this relation becomes,

$$\boldsymbol{\omega} \cdot \mathbf{v} = \mathbf{s}^T Q_0 \mathbf{s} = 0$$

where $Q_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, represents a 4-dimensional quadric in $\mathbb{P}^5$. All lines in $\mathbb{P}^3$, or all lines in $\mathbb{R}^3$ plus lines at infinity (those with $\boldsymbol{\omega} = \mathbf{0}$).

A Line through Successive Points

General rigid motion: rotation followed by translation along same axis, screw motion. Effect on points,

$$\mathbf{p}_2 = R\mathbf{p}_1 + \mathbf{t}$$

Where R is a rotation matrix and $\mathbf{t}$ translation vector.

Lines from all Points

Same for all points in space. Assume that rotation is about z-axis and consider a point a distance d along the x-axis from the rotation axis. Plücker coordinates of the line are,

$$\mathbf{s} = \begin{pmatrix} \mathbf{p}_1 - \mathbf{p}_2 \\ \mathbf{p}_1 \times \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} d(1 - \cos\theta) \\ -d \sin\theta \\ -\theta h/2\pi \\ 0 \\ -d\theta h/2\pi \\ d^2 \sin\theta \end{pmatrix}$$

Where θ is the rotation angle, $\theta h/2\pi$ the pitch of the motion.

The Quadratic Line Complex

Notice that for any rigid motion and any point can always choose coordinates to as above. Invariants don't depend on choice of coordinates. Compute invariants,

$$\mathbf{l}^T Q_\infty \mathbf{s} = -\theta h \lambda / 2\pi, \qquad \mathbf{l}^T Q_0 \mathbf{s} = d^2 \lambda \sin\theta$$

and

$$\mathbf{s}^T Q_\infty \mathbf{s} = 2d^2(1 - \cos\theta) + \theta^2 h^2 / 4\pi^2.$$

Here $\mathbf{l} = \begin{pmatrix} \lambda \mathbf{k} \\ \mathbf{0} \end{pmatrix}$ is the line representing the axis of the motion and the matrix $Q_\infty = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ gives the other rigid-motion invariant.

The Quadratic Line Complex cont.

Now eliminate d to get an equation satisfied by any line $\mathbf{s}$, arising in this way.

$$(\mathbf{l}^T Q_0 \mathbf{s})(\mathbf{l}^T Q_\infty \mathbf{s}) + q\big(\lambda^2(\mathbf{s}^T Q_\infty \mathbf{s}) - (\mathbf{l}^T Q_\infty \mathbf{s})^2\big) = 0.$$

Here $q = h(\theta/2)/2\pi \tan(\theta/2)$, turns up in other places—Parkin+Hunt's 'quatch'.

The Quadratic Line Complex cont.

Now let $\mathbf{l}$ be a general line, $\mathbf{l} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$, replace λ^2 by $\mathbf{l}^T Q_\infty \mathbf{l} = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$, some algebra, the equation becomes,

$$\mathbf{s}^T K \mathbf{s} = 0.$$

where K is a symmetric 6×6 matrix,

$$K = q \begin{pmatrix} -\Omega^2 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Omega V + V\Omega & \Omega^2 + I \\ \Omega^2 + I & 0 \end{pmatrix}.$$

Where,

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix}.$$

The Quadratic Line Complex cont.

Since these are lines, they lie on the Klein quadric, so can take off Q_0 to get,

$$K = q \begin{pmatrix} -\Omega^2 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Omega V + V\Omega & \Omega^2 \\ \Omega^2 & 0 \end{pmatrix}.$$

Intersection of the Klein quadric with another quadric in $\mathbb{P}^5$, gives 3-dimensional family of lines, classically known a quadratic line complex. Much studied object in the past.
Notice: transformations with the same axis and quatch produce the same quadratic complex of lines (new result?).

Cone of Lines through a Point

Well known property of a quadratic line complex. The lines in the complex through any particular point in space form a quadratic cone. To see this let $\mathbf{p}_0$ be some point in space, all lines through $\mathbf{p}_0$ have the form,

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{p}_0 \times \mathbf{u} \end{pmatrix} = \begin{pmatrix} I \\ P_0 \end{pmatrix} \mathbf{u},$$

P_0 the 3×3 anti-symmetric matrix corresponding to $\mathbf{p}_0$, and $\mathbf{u}$ a vector in an arbitrary direction. Substituting into the equation for the complex gives,

$$\mathbf{u}^T \Big(-2q\Omega^2 + \Omega V + V\Omega + \Omega^2 P_0 - P_0\Omega^2 \Big) \mathbf{u} = 0,$$

that is a homogeneous, degree 2 equation in the components of $\mathbf{u}$, a conic curve.

Rigid Motion of the Object

Suppose object undergoes a rigid motion. Which points on the object are unchanged in the image?

Solve this in a moment, but already can see that the projection of such points into the image plane will give a conic curve.

Rigid Motion of the Camera

Here camera undergos rigid motion, or we have 2 identical cameras (eyes). Classical problem (Helmholtz), which points in space have the same coordinates in both cameras?

Clearly the same as previous problem using inverse transform.

The Horopter

Answer to both problems is a twisted cubic curve known as the Horopter. To see that this is a twisted cubic it is simpler to look at the first problem. A point $\mathbf{p}$ will lie on the Horopter if the optical centre $\mathbf{p}_0$ the point $\mathbf{p}$ and its transform $R\mathbf{p} + \mathbf{t}$ are collinear. Best to use homogeneous coordinates here, $\tilde{\mathbf{p}}^T = (x, y, z, w)$. Now the transformed point can be written as a 4×4 matrix,

$$\begin{pmatrix} R\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = G\tilde{\mathbf{p}}.$$

So the condition for $\mathbf{p}$ to lie on the Horopter can be written,

$$\lambda(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}_0) = \mu(G\tilde{\mathbf{p}} - \tilde{\mathbf{p}}_0),$$

where λ and μ are arbitrary paramenters.

The Horopter is a Twisted Cubic Curve

Rearranging this equation gives,

$$(\mu G - \lambda I)\tilde{\mathbf{p}} = (\mu - \lambda)\tilde{\mathbf{p}}_0.$$

Since we are working in $\mathbb{P}^3$ overall multiplicative factors like $(\mu - \lambda)$ or $\det(\mu G - \lambda I)$ are irrelevant. So the solution can be written,

$$\tilde{\mathbf{p}} = \mathrm{Adj}(\mu G - \lambda I)\tilde{\mathbf{p}}_0,$$

where $\mathrm{Adj}()$ denotes the adjugate of the matrix. The elements of the adjugate matrix are the 3×3 cofactors of the original matrix. Hence we can see that x, y, z and w are given by cubic polynomials in the parameters λ and μ. This is what defines a twisted cubic curve.

Cubical Ellipse

All twisted cubic curves are the same up to projective transformation, but if we only allow affine transformations then the different twisted cubic curves are classified by how they meet the plane at infinity $w = 0$ and the circle at infinity $x^2 + y^2 + z^2 = 0$.

The Horopter is very special in this respect, it meets the plane at infinity in 3 points, 1 real and 2 complex conjugate points both lying on the circle at infinity. This makes the Horopter a special cubical ellipse.

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- Still plenty of geometry to study.