Risk Measures on Networks and Expected Utility

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Abstract

In reliability theory projects are usually evaluated in terms of their riskiness, and often decision under risk is intended as the one-shot-type binary choice of accepting or not accepting the risk. In this paper we elaborate on the concept of risk acceptance, and propose a theoretical framework based on network theory. In doing this, we deal with system reliability, where the interconnections among the random quantities involved in the decision process are explicitly taken into account. Furthermore, we explore the conditions to be satisfied for risk-acceptance criteria to be consistent with the axiomatization of standard expected utility theory within the network framework. In accordance with existing literature, we show that a risk evaluation criterion can be meaningful even if it is not consistent with the standard axiomatization of expected utility, once this is suitably reinterpreted in the light of networks. Finally, we provide some illustrative examples.

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1 Introduction

Reliability theory is of paramount relevance in a number of economic, engineering and environmental contexts. In particular, the reliability of a system is strongly connected with its riskiness. A risk is accepted when the associated system is reliable enough or, in other words, when the cost of its avoidance is higher than a predetermined threshold. Essentially, a risk is accepted when, once a criterion for risk evaluation is fixed, the risk level is below a predetermined threshold (see, e.g., Fischhoff et al., 1981; Aven, 2003, 2007). Thus, in order to provide a scientific representation of individual choices on risk acceptance, the formalization of the risk evaluation mechanism is required.

It is widely accepted that individuals are not risk-neutral, and several important theories have been developed on this fact, mostly building upon von Neumann and Morgenstern's (1944) seminal contribution. The basic idea is that realizations of random amounts should be filtered through the so-called utility function, to capture the evidence that risky quantities are evaluated for what they represent for the decision maker, rather than under a purely objective basis. This important framework is formalized into five axioms, which are satisfied under expected utility maximization (see, e.g., Abrahamsen and Aven, 2008). To be self-contained, we have reported the expected utility axiomatization in the Appendix.

Violations of the expected utility theory axioms stand at the core of the debate in decision theory (see, e.g., Loomes, 1991; Katsikopoulos and Gigerenzer, 2008). However, decision rules inconsistent with expected utility theory are not necessarily meaningless. In this respect, a risk acceptance criterion compared to the axiomatization of the expected utility theory has been proposed by Abrahamsen and Aven (2008). These authors face an engineering reliability problem and introduce the *FAR value* as the expected number of fatalities per 100 million exposed hours. From their point of view, the risk should be accepted when the FAR value is less than 10. However, Abrahamsen and Aven (2008) show that the FAR value risk acceptance criterion is not consistent with the independence axiom of expected utility. Their conclusions elaborate on the perspective of the decision maker without philosophically rejecting neither models violating the expected utility axioms nor their utility-based counterparts.

On the same basis of the above-quoted paper, some authors elaborate on the so-called non-linear utility theory, and provide also acceptance criteria grounded on the analysis of the probability distributions of the involved random variables (see, e.g., Geiger, 2002, 2005, 2008, and the references therein).

Criticisms on the setting of *a-priori* criteria for deciding whenever accepting a risk are expressed by Aven and Vinnem (2005), who write:

> "introduction of pre-determined criteria may give the wrong focus — meeting these criteria rather than obtaining overall good and cost/effective solutions and measures."

More specifically, Aven and Vinnem (2005) discuss risk acceptance for environmental protection in the context of fossil fuels extraction and argue that a risk should be properly characterized and evaluated to construct suitable risk acceptance criteria.

In this paper we share the same perspective of Aven and Vinnem (2005), and present a model for risk acceptance taking care of the main characteristics of the problem. In particular, we formalize the concept of risk in the language of network theory, to capture the presence of interactions between the random variables playing a role in the decision process. In doing this, we contribute to the field of literature related to the so-called *systemic risk*, which is much interested in the consequences related to the existence of interconnections among the components of a system (for the paradigmatic application of systemic risk in the banking sector see, e.g., Rochet and Tirole, 1996; Freixas et al., 2000; Bartram et al., 2007; Haldane and May, 2011). More specifically, we are particularly close to Leippold and Vanini (2003), where the authors elaborate on operational risk acceptance in a network analysis framework. Indeed, Leippold and Vanini (2003) argue that there are many risky variables driving the risk acceptance decision, and they present a topological and a stochastic dependence structure.

Here we model the set of *decision variables* — the random variables involved in the risk acceptance problem — and their connections by a specific network, and opportunely define a network measure to evaluate the risk. A numerical threshold for the network measure is then introduced to identify acceptable and not-acceptable risks.[1]

Moreover, following the line traced by Hendon et al. (1994) and Abrahamsen and Aven (2008), we compare the proposed criterion with classical decision theory. With this aim, we adapt and extend the axioms of the expected utility theory to our specific context, arguing that the weights of network's arcs and nodes play an active role in the identification of the preference order.

We also show that a suitable definition of the model terms leads to a criterion which is consistent with the expected utility theory. On the other hand, we argue that inconsistency with respect to the axioms of expected utility does not necessarily imply that the associated risk-acceptance criteria are meaningless.

To develop our arguments, we fully explore the algebraic structure of the set of networks and suitably define the network binary operators of sum and product by scalars.

The rest of the paper is organized as follows: the next Section contains the formal definition of the set of networks, along with the assessment of

[1]For a survey on the theory of networks, we refer the interested reader to Wasserman and Faust (1994), and Scott (2013).

its algebraic structure. Section 3 outlines the risk-acceptance criterion and presents some relevant examples and applications supporting the developed model under a practical point of view. Section 4 presents the rewriting of the expected utility axioms in the language of networks and discusses the consistency of the proposed criterion with expected utility theory. The last Section offers some concluding remarks.

2 The set of networks

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which contains all the random quantities that will be defined in this paper.

We denote by $\mathcal{A}$ the set collecting all the random variables defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The risk acceptance problem is assumed to be identified by the behaviour of some *relevant* random quantities, which are collected in a subset $\mathcal{S}$ of $\mathcal{A}$ defined as:

$$\mathcal{S} = \{X_1, X_2, \dots, X_n\} \subseteq \mathcal{A}. \tag{1}$$

We will refer to the set $\mathcal{S}$ as the *decision set*, where the variables $X_1, X_2, \dots,$ X_n are the *decision variables.*

The decision set $\mathcal{S}$ is here viewed as the set of the nodes (vertices) of a weighted oriented graph. There exist weights both for the arcs and for the nodes of the graph. Specifically, under a pure formal perspective:

- there exists a function $\rho : \mathcal{S} \to \mathbb{R}$ such that $\rho(X_j) = \rho_j$ is the weight of the node X_j, for each $j = 1, 2, \dots, n$;
- we introduce a binary variable for the identification of the arcs of the graph:

$$v(i,j) = \begin{cases} 1, & \text{if there exists the oriented arc connecting } i \text{ and } j, \\ 0, & \text{otherwise;} \end{cases}$$

for each $i, j = 1, 2, \dots, n$. The case of $i = j$ is associated to the presence of a self-connection of the node i with itself;

- a measure for the asymmetric (oriented) connection between each couple of elements of the decision set is introduced, namely: $\delta_{i \to j} \in \mathbb{R}$ is the measure of the oriented arc from X_i to X_j, for each $i, j = 1, 2, \dots, n$, $i \neq j$. The numbers δ's represent the weights related to the n^2 edges, which are given by $n(n-1)$ coupled connections and n self-connections. It is worth noting that the orientation is meaningless when considering the self connections. However, for the sake of simplicity, we adopt also in this case the "oriented notation" and write $\delta_{i \to i}$, for each $i = 1, 2, \dots, n$.

The graph is then fully identified by the knowledge of its nodes and the related functions ρ's, v's and δ's. The resulting quadruple is a network $\mathbf{N}$ as follows:

$$\mathbf{N} = (\mathcal{S}, \rho, v, \delta), \tag{2}$$

where $\rho = \{\rho_j\}_{j=1,2,\dots,n}$, $v = \{v(i,j)\}_{i,j=1,2,\dots,n}$ and $\delta = \{\delta_{i \to j}\}_{i,j=1,2,\dots,n}$.

The space of the networks is then given by

$$\mathbf{Net} = \left\{ (\mathcal{S}, \rho, v, \delta) \in \mathcal{P}(\mathcal{A}) \times \mathbb{R}^{|\mathcal{S}|} \times \{0,1\}^{|\mathcal{S}|^2} \times \mathbb{R}^{|\mathcal{S}|^2} \right\},$$

where $\mathcal{P}(\star)$ indicates the set of the part of $\star$.

The risk problems are then here modeled through specific networks, which are assumed in the present framework to fully describe the investigated risky projects (to be or not to be accepted).

2.1 Algebraic structure of the set Net

This section contains the discussion on some algebraic properties of the set **Net**. In so doing, we pursue three scopes: first, we convince the reader that the space of networks is rather wide, and can be properly used to model a relevant number of risky projects; second, we construct the theoretical framework for introducing a risk-acceptance criterion based on networks; third, we offer the instruments for checking whenever such a risk-acceptance criterion is consistent with the axiomatization of the expected utility theory.

To proceed, we need the definition of the concept of equivalence of networks within the set of the elements of **Net**.

Definition 1. *Consider two networks*

$$\mathbf{N}_k = \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right) \in \mathbf{Net}, \qquad k = 1, 2.$$

We say that $\mathbf{N}_1$ *is equivalent to* $\mathbf{N}_2$ *— and we write* $\mathbf{N}_1 \equiv \mathbf{N}_2$ *— when one of the following conditions is satisfied:*

(i) $\left(\mathcal{S}^{(1)}, \rho^{(1)}, v^{(1)}, \delta^{(1)}\right) = \left(\mathcal{S}^{(2)}, \rho^{(2)}, v^{(2)}, \delta^{(2)}\right)$*;*

(ii) if $1 = v^{(k_1)}(i,j) \neq v^{(k_2)}(i,j) = 0$*, then* $\delta^{(k_1)}_{i \to j} = 0$*, for each* i, j *indices of the nodes of* $\mathcal{S}^{(k_1)}$ *and* $k_1, k_2 = 1, 2$ *with* $k_1 \neq k_2$*, all other things being equal;*

(iii) if $X_j \in \mathcal{S}^{(k_1)} \setminus \mathcal{S}^{(k_2)}$*, then* $\rho^{(k_1)}_j = 0$ *and* $\delta^{(k_1)}_{i \to j} = \delta^{(k_1)}_{j \to i} = 0$*, for each* i *index of the nodes in* $\mathcal{S}^{(k_1)} \cup \mathcal{S}^{(k_2)}$ *and* $k_1, k_2 = 1, 2$ *with* $k_1 \neq k_2$*, all other things being equal.*

The equivalence relation introduced in Definition 1 leads to the identification of equivalence classes in the set **Net**. It assumes that a zero-weighted connection means absence of connection and a node with zero weight can be removed from the decision set. These conditions will turn out to be useful in the definition of the risk measures on **Net**.

Remark 1. *Given a network* $\mathbf{N}$*, it is possible to construct a network* $\widetilde{\mathbf{N}}$ *equivalent to* $\mathbf{N}$ *by adding further nodes* $j_1, \ldots, j_k$ *such that* $\rho_j = 0$ *and* $\delta_{i \to j} = \delta_{j \to i} = 0$*, for each* $j = j_1, \ldots, j_k$ *and for each* i*-th node of* $\widetilde{\mathbf{N}}$*. Hence, under such conditions on the weights of arcs and nodes, it is not restrictive* *to assume that networks share the same set of nodes, which is the maximal one. Moreover, it is also equivalent to consider $v(i,j) = 0$ or, alternatively, $v(i,j) = 1$ and $\delta_{i\to j} = \delta_{j\to i} = 0$. Therefore, under the condition of null weights of not existing arcs, it is not restrictive to assume $v(i,j) = 1$, for each i,j.*

It is worth exploring the topological structure of **Net** by introducing the operators acting on it.

Definition 2. *Consider $\mathbf{N}_1, \mathbf{N}_2 \in$ **Net** such that $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$ for $k = 1, 2$.*

The direct sum of $\mathbf{N}_1$ and $\mathbf{N}_2$ is

$$\mathbf{N}_1 \oplus \mathbf{N}_2 \equiv \mathbf{N}, \tag{3}$$

where $\mathbf{N} \equiv (\mathcal{S}, \rho, v, \delta)$ with $\mathcal{S} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$ and, for each $i,j = 1,\dots,n$, it is $\rho_j = \rho_j^{(1)} + \rho_j^{(2)}$, $\delta_{i\to j} = \delta_{i\to j}^{(1)} + \delta_{i\to j}^{(2)}$ and $v(i,j) = \max\{v^{(1)}(i,j), v^{(2)}(i,j)\}$.

By Definition 2, the direct sum of two networks is equivalent to a new one with all the nodes and the connections existing in the two summed networks. The weights of the new network comes out from aggregating — i.e. summing — those of the summed ones.

We note that the distinction between the sets of nodes $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ and the connection binary variables $v^{(1)}$ and $v^{(2)}$ are useful here to have an intuitive view of the concept of summed networks, even if Remark 1 could suggest to impose $\mathcal{S}^{(1)} = \mathcal{S}^{(2)}$ and $v(i,j) = 1$, for each i,j.

Definition 3. *Consider $\mathbf{N} \equiv (\mathcal{S}, \rho, v, \delta) \in$ **Net** and a scalar $\alpha \in \mathbb{R}$.*

The product scalar-network $\alpha \cdot \mathbf{N}$ is a new network $\mathbf{N}_\alpha \equiv \left(\mathcal{S}, \rho^{(\alpha)}, v, \delta^{(\alpha)}\right)$ such that $\rho_j^{(\alpha)} = \alpha \cdot \rho_j$ and $\delta_{i\to j}^{(\alpha)} = \alpha \cdot \delta_{i\to j}$, for each $i,j = 1,\dots,n$.

Definition 3 explains that a scale factor applied to a network plays the role of applying such a scale factor to the weights of the arcs and the nodes of the network.

The assessment of the topological structure of the set **Net** — endowed with the binary operator $\oplus$ — is of particular relevance for adapting to the network framework the axiomatization of expected utility theory. We formalize this topological framework as follows:

Theorem 1. *The couple* $(\mathbf{Net}, \oplus)$ *is an Abelian group.*

Proof. To prove the theorem, we need to check the validity of the five axioms identifying an Abelian group[2].

G.1 If $\mathbf{N}_1, \mathbf{N}_2 \in \mathbf{Net}$, then $\mathbf{N}_1 \oplus \mathbf{N}_2 \in \mathbf{Net}$.

This is a direct consequence of Definition 2.

G.2 If $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3 \in \mathbf{Net}$, then $\mathbf{N}_1 \oplus [\mathbf{N}_2 \oplus \mathbf{N}_3] \equiv [\mathbf{N}_1 \oplus \mathbf{N}_2] \oplus \mathbf{N}_3$.

In fact, define $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$ for $k = 1, 2, 3$.

Then, this axiom is easily verified, being:

$$\begin{cases} \left(\mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}\right) \cup \mathcal{S}^{(3)} = \mathcal{S}^{(1)} \cup \left(\mathcal{S}^{(2)} \cup \mathcal{S}^{(3)}\right) \; ; \\ \left(\rho_j^{(1)} + \rho_j^{(2)}\right) + \rho_j^{(3)} = \rho_j^{(1)} + \left(\rho_j^{(2)} + \rho_j^{(3)}\right) \quad \forall\, j \; ; \\ \left(\delta_{i\to j}^{(1)} + \delta_{i\to j}^{(2)}\right) + \delta_{i\to j}^{(3)} = \delta_{i\to j}^{(1)} + \left(\delta_{i\to j}^{(2)} + \delta_{i\to j}^{(3)}\right) \quad \forall\, i, j \; ; \\ \max\left[\max\{v^{(1)}(i,j), v^{(2)}(i,j)\}, v^{(3)}(i,j)\right] = \max\left[v^{(1)}(i,j), \right. \\ \qquad\qquad \left. \max\{v^{(2)}(i,j), v^{(3)}(i,j)\}\right] \quad \forall\, i, j \, . \end{cases}$$

G.3 There exists an unique null network $\mathbf{N}_0 \in \mathbf{Net}$ such that $\mathbf{N}_0 \oplus \mathbf{N} \equiv \mathbf{N} \oplus \mathbf{N}_0 \equiv \mathbf{N}$, for each $\mathbf{N} \in \mathbf{Net}$.

In fact, consider $\mathbf{N} \equiv (\mathcal{S}, \rho, v, \delta)$ and define the null network $\mathbf{N}_0 \equiv \left(\mathcal{S}^{(0)}, \rho^{(0)}, v^{(0)}, \delta^{(0)}\right)$ such that:

$$\begin{cases} \mathcal{S}^{(0)} = \mathcal{S} \\ \rho_j^{(0)} = 0 \quad \forall\, j \in \mathcal{S}^{(0)}; \\ \delta_{i\to j}^{(0)} = 0 \quad \forall\, i, j \,|\, v^{(0)}(i,j) = 1; \\ v^{(0)} \quad \text{arbitrary.} \end{cases}$$

[2] For the concept of Abelian group see, e.g., Robinson (1996).

Definition 2 gives that $\mathbf{N} \oplus \mathbf{N}_0 \equiv \mathbf{N}_0 \oplus \mathbf{N} \equiv \mathbf{N}$.

The uniqueness — to be intended in the sense of the equivalence classes introduced in Definition 1 — comes out from Definitions 1 and 2.

G.4 Consider $\mathbf{N} \in \mathbf{Net}$. There exists the inverse network $\mathbf{N}^{-1} \in \mathbf{Net}$ such that $\mathbf{N}^{-1} \oplus \mathbf{N} \equiv \mathbf{N} \oplus \mathbf{N}^{-1} \equiv \mathbf{N}_0$.

In fact, denote $\mathbf{N} \equiv (\mathcal{S}, \rho, v, \delta)$ and define $\mathbf{N}^{-1} \equiv \left(\mathcal{S}^{-1}, \rho^{-1}, v^{-1}, \delta^{-1}\right)$ such that:

$$\begin{cases} \mathcal{S}^{-1} = \mathcal{S} & \\ \rho_j^{-1} = -\rho_j & \forall\, j \in \mathcal{S}^{-1}; \\ v^{-1}(i,j) \geq v(i,j) & \forall\, i,j; \\ \delta_{i\to j}^{-1} = -\delta_{i\to j} & \forall\, i,j \,|\, v^{-1}(i,j) = v(i,j); \\ \delta_{i\to j}^{-1} = 0 & \forall\, i,j \,|\, v^{-1}(i,j) > v(i,j). \end{cases}$$

By Definition 2 we have $\mathbf{N} \oplus \mathbf{N}^{-1} \equiv \mathbf{N}^{-1} \oplus \mathbf{N} \equiv \mathbf{N}_0$.

G.5 The operator $\oplus$ is commutative in $\mathbf{Net}$, i.e. $\mathbf{N}_1 \oplus \mathbf{N}_2 \equiv \mathbf{N}_2 \oplus \mathbf{N}_1$, for each $\mathbf{N}_1, \mathbf{N}_2 \in \mathbf{Net}$.

The validity of this axiom is due to Definition 2, by applying the commutative property of the operator $\cup$ and $+$ over their respective action sets.

Validity of G.1–G.5 proves the Theorem. $\square$

The following result states that the set **Net** is closed with respect to linear combinations, to be intended in the sense of Definitions 2 and 3.

Proposition 1. *Consider* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $\mathbf{N}_1, \mathbf{N}_2 \in \mathbf{Net}$ *such that* $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$ *for* $k = 1, 2$.

Then $\alpha_1 \cdot \mathbf{N}_1 \oplus \alpha_2 \cdot \mathbf{N}_2 \in \mathbf{Net}$.

Proof. The proof is a direct consequence of Theorem 1 and Definition 3. $\square$

A simple and meaningful consequence of Proposition 1 is the following:

Corollary 1. *The set* **Net** *is convex.*

Proof. The proof comes out from Proposition 1, by considering $\alpha_1, \alpha_2 \in [0,1]$, with $\alpha_1 + \alpha_2 = 1$. $\square$

3 Risk acceptance criterion

A general concept of risk measure on networks is now introduced. More details and illustrative examples will be provided below.

Definition 4. *A risk measure on networks (RMN, hereafter) is a function* $\mu : \mathbf{Net} \to \mathbb{R}$ *such that:*

- *If* $\mathbf{N}_1 \equiv \mathbf{N}_2$*, then* $\mu(\mathbf{N}_1) = \mu(\mathbf{N}_2)$*;*
- *the function* μ *induces a total order* $\preceq_\mu$ *in* **Net** *as follows:*

$$\mathbf{N}_1 \preceq_\mu \mathbf{N}_2 \qquad \textit{if and only if} \qquad \mu(\mathbf{N}_1) \leq \mu(\mathbf{N}_2),$$

  for each $\mathbf{N}_1, \mathbf{N}_2 \in \mathbf{Net}$*;*

- $\mu(\mathbf{N}_1) < \mu(\mathbf{N}_2)$ *means that* $\mathbf{N}_1$ *is less risky — in the sense captured by the preference order* $\preceq_\mu$ *— than* $\mathbf{N}_2$*.*

We collect the RMNs in a set $\mathbf{M}_{\mathbf{Net}}$.

Definition 4 is rather general. It simply states that any function which respects the equivalence between networks — in the sense that it assigns an identical value to equivalent networks — and which induces a preference order describing riskiness over **Net** is a RMN. Of course, each risk can be viewed as a network belonging to a specific subset of $\mathbf{Net}_\star \subseteq \mathbf{Net}$ which contains networks with peculiar characteristics. In this respect, Definition 4 can be rewritten by considering a restriction of μ to an opportunely defined

$\mathbf{Net}_\star$ which induces a total order on it. To assist the reader in grasping this point, an example is provided below.

The uniqueness of the RMN is also a relevant theme, and it is quite simple to see that networks can be identically ordered by employing different μ's. The following Definition is particularly useful:

Definition 5. *Consider two RMNs* $\mu_k : \mathbf{Net} \to \mathbb{R}$*, with* $k = 1, 2$.

μ_1 *is equivalent to* μ_2 *— and we indicate* $\mu_1 \equiv \mu_2$ *— if and only if*

$$\mu_1(\mathbf{N}_1) \leq \mu_1(\mathbf{N}_2) \qquad \text{if and only if} \qquad \mu_2(\mathbf{N}_1) \leq \mu_2(\mathbf{N}_2),$$

for each $\mathbf{N}_1, \mathbf{N}_2 \in \mathbf{Net}$.

Definition 5 leads to the identification of equivalence classes over the set $\mathbf{M}_{\mathbf{Net}}$.

The following example contributes to the understanding of the points illustrated above:

Example 1. *A financial institution offers to Frank and Lisa the same structured product which can be replicated by a portfolio composed by units of three risky assets with stochastic uniperiodal returns* X_1*,* X_2 *and* X_3*. The shares of capital forming the replication portfolio are* (x_1, x_2, x_3)*,being* x_j *the percentage of the capital invested in the asset with return* X_j*, for* $j = 1, 2, 3$*. It is assumed that* $x_1 + x_2 + x_3 = 1$*, i.e. the entire capital is invested. Moreover,* $x_j \in [0, 1]$*, i.e. short selling is not allowed. In a uniperiodal world, the investor must take a decision by guessing what will happen at time 1.*

Frank evaluates the riskiness of the product by computing the maximum value of the Pearson correlation coefficients between couples of returns multiplied with the related shares of portfolio. As the value of the maximum value of the Pearson correlation coefficient increases, the riskiness of the corresponding portfolio grows.

In this case, each portfolio can be represented through a (symmetric) network $\mathbf{N}_F$*, where the subscript* F *means "related to Frank". The subset* *of* **Net** *containing such portfolios is denoted by* $\mathbf{Net}_F$ *and, evidently, it is not unique. A meaningful definition of* $\mathbf{Net}_F$ *is the following:*

- *the decision variables — i.e. the vertices of each network* $\mathbf{N}_F \in \mathbf{Net}_F$ *— are fixed, and given by* X_1*,* X_2 *and* X_3*;*
- *the weights of the (oriented) arcs are fixed, and given by:*

  $$\delta_{i\to j} = \delta_{j\to i} = \frac{\mathbb{C}\left[X_i, X_j\right]}{\sqrt{\mathbb{V}\left[X_i\right]} \cdot \sqrt{\mathbb{V}\left[X_j\right]}}, \qquad i, j = 1, 2, 3,$$

  where $\mathbb{C}$ *and* $\mathbb{V}$ *are the covariance and variance operator, respectively;*
- *since, by definition of the correlation coefficient, there exists a connection between each couple of different nodes (self-connection and unitary correlation coefficient is not introduced in Frank's decision process), then* $v(i, j) = 1$ *for each* $i, j = 1, 2, 3$*,* $i \neq j$*, and* $v(i, i) = 0$*, for each* $i = 1, 2, 3$*;*
- *functions* ρ*'s are given by the shares of portfolio, so that* $\rho_j = x_j$*, for* $j = 1, 2, 3$*. Hence,* $\rho \in [0, 1]^3$*.*

The RMN adopted by Frank is (equivalent to) $\mu_F : \mathbf{Net}_F \to [-1, 1]$ *such that:*

$$\mu_F(\mathbf{N}_F) \equiv \max_{i,j=1,2,3} v(i, j)\rho_i\rho_j\delta_{i\to j}. \tag{4}$$

Lisa has a different strategy. She takes X_1 *as reference return, and evaluates the risk of a portfolio by making the difference between the probabilities that* $x_1 \cdot X_1$ *is greater than* $x_2 \cdot X_2$ *and that it is greater than* $x_3 \cdot X_3$*. Such a difference increases with the riskiness of the portfolio.*

Portfolios can be modeled through networks $\mathbf{N}_L \in \mathbf{Net}_L \subseteq \mathbf{Net}$*, where the subscript L means "referred to Lisa". A meaningful definition of the subset* $\mathbf{Net}_L$ *is the following:*

- *the vertices of the networks are, also in this case,* X_1*,* X_2 *and* X_3*;*

- *the weights of the (oriented) arcs are given by:*

$$\delta_{i\to j} = \mathbb{P}\left[x_i \cdot X_i > x_j \cdot X_j\right], \qquad i=1,\, j=2,3. \tag{5}$$

  Hence,

$$\delta_{i\to j} \in \begin{cases} [0,1], & \textit{if } i=1 \textit{ and } j=2,3; \\ \mathbb{R}, & \textit{otherwise;} \end{cases}$$

- *the binary variables in v are given by*

$$v(i,j) = \begin{cases} 1, & \textit{if } (i,j) \in \{(1,2),(1,3)\} \\ 0, & \textit{otherwise.} \end{cases}$$

- $\rho \in \mathbb{R}^3$.

The RMN adopted by Lisa is (equivalent to) $\mu_L : \mathbf{Net}_L \to [-1,1]$ *such that:*

$$\mu_L(\mathbf{N}_L) \equiv \delta_{1\to 3} - \delta_{1\to 2}. \tag{6}$$

Some comments on Example 1 are in order: Frank and Lisa's decision processes proceed through the analysis of the weights of the arcs and of the nodes. In particular, Frank considers a specific subfamily of networks $\mathbf{Net}_F \subset \mathbf{Net}$, with nodes given by the stochastic returns X_1, X_2, X_3, the δ's given by the correlation coefficients and $v(i,j) = 1$ if and only if $i \neq j$, and all the possible nonnegative weights ρ's for the nodes whose sum is unitary. $\mathbf{Net}_F$ collects the available portfolios obtained by the assets with returns X_1, X_2 and X_3 and correlation coefficients captured by the vector δ.

By construction, Frank refuses the risk in the case of a replicating portfolio of *polarized* type, i.e. in presence of a couple of nodes (X_i, X_j) with a high value of ρ_i and ρ_j and also a great level of connection $\delta_{i\to j}$. Polarized portfolios are those where the main part of the capital is shared among the returns with a high correlation coefficients, probably positively correlated. The motivation for rejecting such a risk could be found in the attitude towards the risk of Frank, who seems to be risk-averse. In fact, the polarized portfolios defined above are, in the general theory of finance, those more risky, even if they provide a great level of expected return.

As far as Lisa's perspective is concerned, she considers a subset of networks $\mathbf{Net}_L \subset \mathbf{Net}$, with nodes given by the stochastic returns X_1, X_2, X_3, $v(i,j) = 1$ if and only if $(i,j) \in \{(1,2),(1,3)\}$, the δ's belonging to $[0,1]$ and all the possible nonnegative weights ρ's for the nodes whose sum is unitary. As in the Frank's case, the set $\mathbf{Net}_L$ collects the available portfolios obtained by the assets with returns X_1, X_2 and X_3. However, on the basis of the different definition of the weights of nodes and arcs, we have $\mathbf{Net}_L \neq \mathbf{Net}_F$. Moreover, it is interesting to note that the network measure $\mu_L(\mathbf{N}_L)$ increases when the realizations of X_2 grow and those of X_3 decrease, letting the other quantities be the same. This is a direct consequence of definition (6) and of the remark that, in this case, $\mathbb{P}[X_1 > X_2]$ decreases and $\mathbb{P}[X_1 > X_3]$ increases. More formally, fix a portfolio (x_1, x_2, x_3) and construct two networks

$$\mathbf{N}_L^{(a)} \equiv \left(\{X_1, X_2^{(a)}, X_3^{(a)}\}, \rho, v, \delta^{(a)}\right), \quad \mathbf{N}_L^{(b)} \equiv \left(\{X_1, X_2^{(b)}, X_3^{(b)}\}, \rho, v, \delta^{(b)}\right),$$

where $X_2^{(a)} >_{SD1} X_2^{(b)}$, $X_3^{(b)} >_{SD1} X_3^{(a)}$ and the δ's are defined according to (5). Then $\mu_L(\mathbf{N}_L^{(a)}) \geq \mu_L(\mathbf{N}_L^{(b)})$.

From a behavioural finance point of view, return X_1 represents an *anchor* for Lisa's decision process. The assessment of how rational is the selection of such an anchor is beyond the scopes of the present paper, but it is clear that X_1 can be a financially reasonable reference point or, simply, the effect of a psychological bias (see Ritter, 2003).

Example 1 suggests also the presence of a couple of thresholds $H_F, H_L \in [-1,1]$ such that Frank (Lisa) purchases the portfolio $\mathbf{N}_F \in \mathbf{Net}_F$ ($\mathbf{N}_L \in \mathbf{Net}_L$) if and only if $\mu_F(\mathbf{N}_F) \leq H_F$ ($\mu_L(\mathbf{N}_L) \leq H_L$).

We are now in the position of defining the risk acceptance criterion.

Definition 6. *Consider a RMN $\mu \in \mathbf{M}_{\mathbf{Net}}$ and a constant $H \in \mathbb{R}$. More-*

over, consider a risk modeled through a network $\mathbf{N} \in \mathbf{Net}$.

Network $\mathbf{N}$ *is said to be an acceptable risk at level* H *if and only if*

$$\mu(\mathbf{N}) \leq H. \tag{7}$$

Definition 6 is in accord to the concept of RMN introduced in Definition 4. In fact, Definition 6 says implicitly that a great value of $\mu(\mathbf{N})$ means a high level of risk for the system described by the network $\mathbf{N}$.

3.1 A real world application: insurability of nuclear risks

Insurability is a concept related to a specific risk, and represents the description of the conditions under which an insurance company has economic convenience to stipulate a contract for insuring such a risk. It is clear that the problem of insurability is of great relevance when the event associated to the insured risk leads to losses of large amount, i.e. in presence of the so-called *catastrophic events*.

In the context of catastrophic events, a prominent role is played by the nuclear risk. The problem of insurability of a risk in the nuclear case is thus of particular interest. This task is widely discussed at an academic as well as institutional level, and we refer to the authoritative scientific documentation provided by Faure and Hartlief (2003). Nuclear risk is commonly acknowledged to be a "systemic" one, in that the occurrence of a nuclear disaster has a dramatic impact on a number of human activities and environmental entities. Hence, the evaluation of the damages generated by a nuclear accident can be performed only by analyzing the network describing all the entities involved in the disaster, along with their interconnections and their individual relative relevance in the overall context. Therefore we assume the existence of a network $\mathbf{N}_{nuclear} = (\mathcal{S}, \rho, v, \delta)$ associated to the nuclear risk.

Faure and Hartlief (2003) explain that major insurance companies of any nuclear country operate on a cooperative and non-competitive basis and pool their resources to jointly insure the nuclear risk. This fact implicitly means that each individual insurance company does not accept the risk, whereas is willing to accept it only when the riskiness of the nuclear accident is suitably reduced. This behaviour can be easily formalized in the language used in this paper.

Indeed, consider a country with K (major) insurance companies. Assume that the k-th company measures the nuclear risk through a RMN μ_k, and accepts the risk if and only if the risk measure is below a specific threshold $\bar{H}_k \in \mathbb{R}$, for each $k = 1, 2, \ldots, K$. The evidence that individual companies do not accept the risk can be interpreted as the existence of a real number $H_k \in \mathbb{R}$ such that

$$\mu_k\left(\mathbf{N}_{nuclear}\right) = H_k > \bar{H}_k,$$

for each $k = 1, 2, \ldots, N$.

The pooling strategy acts by sharing the overall risk among all the K insurance companies. Formally, the interaction among companies leads to a transformation of the network $\mathbf{N}_{nuclear}$ into the new one $\widetilde{\mathbf{N}}_{nuclear}$, with the convenience that there exist K weights $\alpha_1, \alpha_2, \ldots, \alpha_K \in (0, 1)$ such that

$$\mu_k\left(\widetilde{\mathbf{N}}_{nuclear}\right) = \alpha_k \cdot H_k,$$

and $\alpha_k \cdot H_k < \bar{H}_k$.

Hence, the cooperation reduces the nuclear risk level, and such a reduced risk is then accepted by all the insurance companies.

4 Consistency of RMNs with the expected utility theory

The link between utility theory and risk criteria can be easily identified: expected utility is maximized by the decision maker while the risk — captured by the function μ — is minimized by the decider (and accepted only if its level is below a certain threshold, see Definition 6).

This section contains a proposal for rewriting the classical axioms of the expected utility theory in terms of the networks belonging to the set **Net** and of the RMNs μ.

In general the realizations of the decision variables may be less risky when they are great or, depending on the specific considered model, low. In accordance with this observation, generally there is no direct relation between high (low) levels of weights δ's and connections ρ's with high (low) risk of the associated network. Therefore, it is not possible to propose an *a-priori* universal view of the networks as risk models. However, it is still possible to construct a theoretical framework for providing the conditions for consistency of the network risk-acceptance criterion with standard expected utility theory.

To this end, it is important to point out that the original axioms of expected utility theory listed in the introduction are based on a "positive role" played by high probabilities over high realizations on the preference order. An illustrative example of this fact can be found in axiom A.3, where it is clear that higher probabilities of higher realizations are preferred. This is of course a conventional agreement needed for stating an axiomatization of the expected utility theory: however, it is not the unique way to read the concept of lottery. In fact, lotteries X and Y could be *a-priori* defined as the amount to be lost by the player. In this case, axiom A.3 should be reverted to maintain reasonableness in decision theory.

We here move from an analogous basis. In particular, we prepare — when needed — the theoretical ground for deriving the conditions on μ such that the risk-acceptance criterion of Definition 6 fits with the axioms of expected utility theory.

First, we provide a reformulation of axioms A.1–A.5 in the language of networks, to include also the role of the weights (see axioms A.1′–A.5′).

Axioms A.1′ and A.2′ are simple rewritings of the corresponding axioms

A.1 and A.2 by replacing "lotteries" with "networks" and by using the operators introduced in Definitions 2 and 3.

A.1′ Weak order — Preferences are: 1) *complete*, i.e. the decider can state whether two networks are equivalent or whether one is preferred to the other; 2) *transitive*, i.e. given three networks $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$, if $\mathbf{N}_1$ is preferred to $\mathbf{N}_2$ and $\mathbf{N}_2$ is preferred to $\mathbf{N}_3$, then $\mathbf{N}_1$ is preferred to $\mathbf{N}_3$; 3) *reflexive*, i.e. the decision-maker is indifferent between two equivalent networks.

No requirements are needed to let Axiom A.1′ be satisfied.

Proposition 2. *Consider $\mu \in \mathbf{M}_{\mathbf{Net}}$ and the induced order preference $\preceq_\mu$. Then, $\preceq_\mu$ satisfies Axiom A.1′.*

Proof. We discuss separately the three properties leading to the weak order axiom.

1) Completeness is due to the property of the RMN of inducing a total order over the set **Net**.

2) Transitivity comes out easily from Definition 4, being the preference order $\preceq_\mu$ over **Net** induced — by means of the real function μ — by the preference order $\leq$ over $\mathbb{R}$.

3) Reflexivity is a direct consequence of Definition 4, for which if $\mathbf{N}_1 \equiv \mathbf{N}_2$ then $\mu(\mathbf{N}_1) = \mu(\mathbf{N}_2)$, for each $\mathbf{N}_1, \mathbf{N}_2 \in \mathbf{Net}$. □

A.2′ Continuity — Given three different lotteries $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ such that $\mathbf{N}_1$ is preferred to $\mathbf{N}_2$ and $\mathbf{N}_2$ is preferred to $\mathbf{N}_3$, then there exists a number $p \in (0, 1]$ such that the decider is indifferent between the compounded lottery $p \cdot \mathbf{N}_1 \oplus (1 - p) \cdot \mathbf{N}_3$ and $\mathbf{N}_2$.

Some conditions are needed to let the preference order be in line to Axiom A.2′.

Proposition 3. *Assume that* $\mu \in \mathbf{M_{Net}}$ *satisfies one of the following conditions:*

(i) μ *is independent from the weights* δ*'s and it is linear with respect to* ρ_j*, for each* $j = 1, \dots, n$*;*

(ii) μ *is independent from the weights* ρ*'s and it is linear with respect to* $\delta_{i \to j}$*, for each* $i, j = 1, \dots, n$*.*

Then the induced order preference $\preceq_\mu$ *satisfies Axiom A.2'.*

Proof. Consider three networks $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3 \in \mathbf{Net}$ such that $\mathbf{N}_1 \preceq_\mu \mathbf{N}_2$ and $\mathbf{N}_2 \preceq_\mu \mathbf{N}_3$. Then we can write

$$\mu(\mathbf{N}_1) \leq \mu(\mathbf{N}_2) \leq \mu(\mathbf{N}_3),$$

which means that there exists $p \in [0, 1]$ such that

$$\mu(\mathbf{N}_2) = p\mu(\mathbf{N}_1) + (1 - p)\mu(\mathbf{N}_3).$$

To prove the result, we need to check that

$$\mu(\mathbf{N}) = p\mu(\mathbf{N}_1) + (1 - p)\mu(\mathbf{N}_3),$$

with

$$\mathbf{N} \equiv p \cdot \mathbf{N}_1 \oplus (1 - p) \cdot \mathbf{N}_3. \tag{8}$$

Under Remark 1, define the networks as follows: $\mathbf{N}_k \equiv \left(\mathcal{S}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$, for $k = 1, 2, 3$. Then, Definitions 3 and 2 and formula (8) lead to $\mathbf{N} \equiv (\mathcal{S}, \rho, v, \delta)$, with

$$\begin{cases} \rho_j = p\rho_j^{(1)} + (1 - p)\rho_j^{(3)}; & \forall\, j \\ \delta_{i \to j} = p\delta_{i \to j}^{(1)} + (1 - p)\delta_{i \to j}^{(3)}, & \forall\, i, j \\ v(i, j) = \max\{v(i, j)^{(1)}, v(i, j)^{(3)}\}, & \forall\, i, j. \end{cases} \tag{9}$$

Hence, by assuming that one among conditions (i) and (ii) holds, relations in (9) give the thesis. □

A.3′ Preference increasing with probabilities (and with connections and weights) — Consider $\mathcal{S}^{(1)} = \{X_1^{(1)}, X_2\}$ and $\mathcal{S}^{(2)} = \{X_1^{(2)}, X_2\}$, where $X_1^{(1)}, X_1^{(2)}, X_2 \in \mathcal{A}$ and $X_1^{(1)} >_{SD1} X_2^{(2)}$, where $>_{SD1}$ denotes stochastic dominance of order 1.

Moreover, consider two networks $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v, \delta^{(k)}\right)$, for $k = 1, 2$, such that:

$$\begin{cases} \rho_1^{(1)} \geq \rho_1^{(2)}; \\ \rho_2^{(1)} = \rho_2^{(2)}; \\ \delta_{i \to j}^{(1)} \geq \delta_{i \to j}^{(2)}, \quad \text{for } i = 1,\, j = 2; \\ \delta_{i \to j}^{(1)} = \delta_{i \to j}^{(2)}, \quad \text{otherwise.} \end{cases} \tag{10}$$

Then the decision maker prefers network $\mathbf{N}_1$ to $\mathbf{N}_2$.

Axiom A.3′ extends and complements axiom A.3 in expected utility theory. In fact, consider the following scheme: (i) system (10) is verified with all equalities, and some zero weights are taken to remove the node X_2 in accord to Definition 1; (ii) the variables $X_1^{(1)}$ and $X_1^{(2)}$ may assume only two realizations a, b, with $a < b$; (iii) it results $P(X_1^{(1)} = a) < P(X_1^{(2)} = a)$. Then axiom A.3 means that $X_1^{(1)}$ is preferred to $X_1^{(2)}$. Moreover, we have $X_1^{(1)} >_{SD1} X_1^{(2)}$, and this leads also to axiom A.3′, so that $\mathbf{N}_1$ is better than $\mathbf{N}_2$.

Axiom A.3′ states also that the decider is assumed to prefer high values of weights and connections for the nodes with better performance (to be here intended in the general probabilistic sense of the stochastic dominance). Of course, axiom A.3′ can be extended to the general case of n decision variables and in presence of couples of them ordered according to stochastic dominance of order 1.

To let the preference order $\preceq_\mu$ be consistent with axiom A.3′, the RMN μ should satisfy a natural requirement:

Proposition 4. *Consider the networks* $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v, \delta^{(k)}\right)$*, for* $k = 1, 2$ *as in the statement of axiom A.3′, with* $X_1^{(1)} >_{SD1} X_2^{(2)}$ *and assuming* *that (10) holds.*

The order preference $\preceq_\mu$ satisfies axiom A.3′ if and only if $\mu(\mathbf{N}_1) < \mu(\mathbf{N}_2)$.

Proof. By construction of the networks $\mathbf{N}_1$ and $\mathbf{N}_2$ and by the formalization of the axiom A.3′, we obtain the thesis. □

A.4′ Compound networks — Fix $n \in \mathbb{N}$. Consider n networks $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$, for $k = 1, 2, \dots, n$, and a compound network $\mathbf{N}^\star$ having the n networks as nodes as follows:

$$\mathbf{N}^\star \equiv (\{\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_n\}, \rho^\star, v^\star, \delta^\star)\,.$$

Furthermore, define $\widetilde{\mathbf{N}} \in \mathbf{Net}$ as $\widetilde{\mathbf{N}} \equiv \left(\widetilde{\mathcal{S}}, \tilde{\rho}, \tilde{v}, \tilde{\delta}\right)$, where

$$\widetilde{\mathcal{S}} = \bigcup_{k=1}^{n} \mathcal{S}^{(k)},$$

two transformations ϕ and ψ can be identified such that:

$$\begin{cases} \tilde{\rho} = \phi\left(\rho^\star, \rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}\right); \\ \tilde{\delta} = \psi\left(\delta^\star, \delta^{(1)}, \delta^{(2)}, \dots, \delta^{(n)}\right) \end{cases} \tag{11}$$

and, for each $(i,j) \in \mathcal{S}^{(k_i)} \times \mathcal{S}^{(k_j)}$, we have

$$\begin{cases} \tilde{v}(i,j) = 1 & \text{if } k_i \neq k_j \text{ and } v^\star(\mathbf{N}_i, \mathbf{N}_j) = 1; \\ \tilde{v}(i,j) = 1 & \text{if } k_i = k_j = k \text{ and } v^{(k)}(i,j) = 1; \\ \tilde{v}(i,j) = 0 & \text{otherwise.} \end{cases}$$

Then, two transformations ϕ and ψ as in (11) exist such that $\widetilde{\mathbf{N}}$ and $\mathbf{N}^\star$ are equivalent for the preference order.

It is evident that axiom A.4′ represents an extension to the networks of the corresponding axiom A.4 of expected utility theory. In fact, axiom A.4 states the deciders reduce a multistage lottery — whose outcomes are lotteries — to a suitably defined one-stage lottery.

We now formalize the conditions to be satisfied by μ in order to let the preference order $\preceq_\mu$ be consistent with axiom A.4′.

Proposition 5. *Fix $n \in \mathbb{N}$, consider n networks $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n$ and a network $\mathbf{N}^\star$ having the n networks as nodes, as in the statement of axiom A.4′.*

Then the order preference $\preceq_\mu$ satisfies axiom A.4′ if and only if two transformations ϕ and ψ as in (11) exist such that:

$$\mu\left(\left(\{\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n\}, \rho^\star, v^\star, \delta^\star\right)\right) = \mu\left((\widetilde{\mathcal{S}}, \tilde{\rho}, \tilde{v}, \tilde{\delta})\right).$$

Proof. The proof is a direct consequence of the formalization of axiom A.4′.

□

A.5′ Independence — Consider two networks $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$, for $k = 1, 2$. Assume that $\mathcal{S}^{(1)} = \{X_1^{(1)}, \ldots, X_n^{(1)}\}$ and $\mathcal{S}^{(2)} = \{X_1^{(2)}, \ldots, X_n^{(2)}\}$, with $X_i^{(1)} \neq X_i^{(2)}$ for each $i = 1, \ldots, n$. Suppose that $X_1^{(1)} >_{SD1} X_1^{(2)}$ and suppose also that $\mathbf{N}_1$ is preferred to $\mathbf{N}_2$. Let us now consider two networks $\widetilde{\mathbf{N}}_1, \widetilde{\mathbf{N}}_2$ defined as follows:

$$\widetilde{\mathbf{N}}_k \equiv \left(\tilde{\mathcal{S}}^{(k)}, \rho^{(k)}, v, \delta^{(k)}\right), \qquad k = 1, 2,$$

where $\tilde{\mathcal{S}}^{(1)} = \{\tilde{X}_1^{(1)}, X_2^{(1)}, \ldots, \tilde{X}_n\}$ and $\tilde{\mathcal{S}}^{(2)} = \{\tilde{X}_1^{(2)}, X_2^{(2)}, \ldots, \tilde{X}_n\}$, being $\tilde{X}_1^{(1)} >_{SD1} \tilde{X}_1^{(2)}$.

Then $\widetilde{\mathbf{N}}_1$ is preferred to $\widetilde{\mathbf{N}}_2$.

Axiom A.5′ provides an extension of axiom A.5 in expected utility theory. In fact, let us impose the necessary zero weights ρ's and δ's to restrict to the case of network $\mathbf{N}_k$ with $\mathcal{S}^{(k)} = \{X_1^{(k)}\}$, for each $k = 1, 2$, in accord to Definition 1. Moreover, assume that $X_1^{(1)}$ and $X_1^{(2)}$ share a common outcome $a \in \mathbb{R}$ and $X_1^{(1)} >_{SD1} X_1^{(2)}$. Now consider $\tilde{X}_1^{(k)}$ as the random variable obtained by replacing in $X_1^{(k)}$ the outcome a with $\tilde{a} \in \mathbb{R}$, so that the new network $\widetilde{\mathbf{N}}_k$ has set of nodes $\tilde{\mathcal{S}}^{(k)} = \{\tilde{X}_1^{(k)}\}$, for $k = 1, 2$. The independence axiom A.5′ is satisfied when the selection of $X_1^{(1)}$ and $X_1^{(2)}$ leads to $\tilde{X}_1^{(1)} >_{SD1} \tilde{X}_1^{(2)}$, and this implies in turn that $\widetilde{\mathbf{N}}_1$ is preferred to $\widetilde{\mathbf{N}}_2$, for each $\tilde{a} \in \mathbb{R}$.

Next Proposition explains the conditions to be satisfied by μ in order to have $\preceq_\mu$ satisfying axiom A.5$'$.

Proposition 6. *Consider four networks* $\mathbf{N}_1, \mathbf{N}_2, \widetilde{\mathbf{N}}_1$ *and* $\widetilde{\mathbf{N}}_2$ *as in the statement of axiom A.5$'$.*
Then the order preference $\preceq_\mu$ *satisfies axiom A.5$'$ if and only if* $\mu(\mathbf{N}_1) < \mu(\mathbf{N}_2)$ *implies* $\mu(\widetilde{\mathbf{N}}_1) < \mu(\widetilde{\mathbf{N}}_2)$.

Proof. The proof stems from the formulation of axiom A.5$'$. □

Let us go back to Example 1, in order to show that risk-acceptance criteria can be reasonably identified even when they do not satisfy the axiomatizations of the expected utility theory in the language of the networks. With this aim, we limit the attention to the mathematical definition of the RMNs μ_F and μ_L as introduced in (4) and (6), respectively.

Proposition 7. *Consider the RMNs* $\mu_F : \mathbf{Net} \to \mathbb{R}$ *and* $\mu_L : \mathbf{Net} \to \mathbb{R}$ *in (4) and (6), respectively. Denote as* $\preceq_F$ *and* $\preceq_L$ *the preference orders induced by* μ_F *and* μ_L*, respectively.*

(*i*) $\preceq_F$ *is not consistent with the expected utility theory;*

(*ii*) $\preceq_L$ *satisfies axioms A.1, A.2, A.3$'$, A.4$'$, A.5$'$, hence being consistent with the axiomatization of the expected utility theory in the language of the networks.*

Proof. We discuss separately the two cases.

(*i*) By (4), it is easily obtained that μ_F is not linear neither with respect to $\delta_{i\to j}$ nor ρ_j, for the presence of the max operator. Therefore, $\preceq_F$ does not satisfy axiom A.2$'$, and so it is not consistent with the expected utility theory.

(*ii*) Propositions 2 and 3 guarantee that $\preceq_L$ satisfies axioms A.1$'$ and A.2$'$.

Formula (6) assures that μ_L is linear with respect to $\delta_{i\to j}$ and independent from the weights ρ's, hence leading to the fulfillment of axiom A.3′.

Consider the networks $\widetilde{\mathbf{N}}$ and $\mathbf{N}^\star$ defined as in the statement of axiom A.4′. Assume that the transformation ψ of (11) is such that $\tilde{\delta}_{i\to j} = \delta^\star_{i\to j}$, for $i = 1$ and $j = 2, 3$. Then $\mu_L(\mathbf{N}^\star) = \mu_L(\widetilde{\mathbf{N}})$, and this means that axiom A.4′ is satisfied by $\preceq_L$.

Consider two networks $\mathbf{N}_k \equiv \left(\mathcal{S}^{(k)}, \rho^{(k)}, v^{(k)}, \delta^{(k)}\right)$, with $k = 1, 2$, such that $\mu_L(\mathbf{N}_1) < \mu_L(\mathbf{N}_2)$. Moreover, consider the networks $\widetilde{\mathbf{N}}_k$, with $k = 1, 2$, as in the statement of axiom A.5′. Since the networks $\mathbf{N}_k$ and $\widetilde{\mathbf{N}}_k$ share the same set of δ's, then we have that $\mu_L(\widetilde{\mathbf{N}}_1) < \mu_L(\widetilde{\mathbf{N}}_2)$ and axiom A.5′ holds for $\preceq_L$.

□

Example 1 and Proposition 7 confirm the conclusion advocated in Abrahamsen and Aven (2008) that the use of risk-acceptance criteria is not necessarily consistent with the expected utility theory, reformulated here in the language of networks.

5 Conclusions

In this paper we discuss the concept of risk-acceptance, when risks are modelled through networks. We propose a description of the algebraic structure of the set of networks, in order to better formalize the theoretical framework we deal with. With this aim, we offer a reformulation of some of the axioms of expected utility theory, extended in the language of networks. Moreover, we show that risk measures — and risk-acceptance criteria — can be properly selected to achieve a consistency condition of the related preference order with the suitable rewriting of the usual expected utility axiomatization. However, it is important to note that inconsistency with standard expected utility theory does not necessarily lead to meaningless risk-acceptance criteria: in this sense we agree with Abrahamsen and Aven (2008).

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Appendix

In the present Appendix we recall the expected utility axiomatization, as in Abrahamsen and Aven (2008).

A.1 Weak order — Preferences are: 1) *complete*, i.e. the decider can state whether two lotteries are equivalent or whether one is preferred to the other; 2) *transitive*, i.e. given three lotteries X, Y, Z, if X is preferred to Y and Y is preferred to Z, then X is preferred to Z; 3) *reflexive*, i.e. the decision-maker is indifferent between two similar lotteries.

A.2 Continuity — Given three different lotteries X, Y, Z such that X is preferred to Y and Y is preferred to Z, then there exists a number $p \in (0, 1]$ such that the decider is indifferent between the compounded lottery $pX + (1 - p)Z$ and Y.

A.3 Preferences increasing with probability — Consider two lotteries X and Y with only two outcomes a and b, where $a > b$. The decider prefers X to Y if and only if $P(X = a) > P(Y = a)$.

A.4 Compound probabilities — Consider a compound lottery X whose outcomes are two lotteries X_1 (with probability p) and X_2 (with probability $1-p$). Then lottery X is indifferent to the simple lottery given by the outcomes of X_1 and X_2 with probabilities obtained by the product of those of the outcomes of X_1 with p (those of the outcomes of X_2 with $1 - p$).

A.5 Independence — Consider two lotteries X and Y sharing a common outcome a and suppose that the decider prefers X to Y. If the lotteries $\widetilde{X}$ and $\widetilde{Y}$ are obtained by replacing a with $\tilde{a}$ in X and Y, respectively, then the decider prefers $\widetilde{X}$ to $\widetilde{Y}$.