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THE SKEW NORMAL MULTIVARIATE RISK MEASUREMENT FRAMEWORK
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Abstract:	In this paper, we consider a random vector X = (X_1;X_2) following a multivariate Skew Normal distribution and we provide an explicit formula for the expected value of X conditioned to the event X <= X*, with X* in (-infty,+infty)^2. Such a conditional expectation has an intuitive interpretation in the context of risk measures.
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THE SKEW NORMAL MULTIVARIATE RISK MEASUREMENT FRAMEWORK

MAURO BERNARDI, ROY CERQUETI, AND ARSEN PALESTINI

ABSTRACT. In this paper, we consider a random vector $X = (X_1, X_2)$ following a multivariate Skew Normal distribution and we provide an explicit formula for the expected value of X conditioned to the event $X \leq \overline{X}$, with $\overline{X} \in \mathbb{R}^2$. Such a conditional expectation has an intuitive interpretation in the context of risk measures.

Keywords: Conditional risk measures; Skew Normal distribution; Value–at–Risk; Expected shortfall.
AMS MSC 2010: 60E05, 62E15, 91G70.

1. INTRODUCTION

The employment of nonstandard probability distributions in financial risk theory represents a growing field of research, leading to either theoretical additions as well as relevant practical implications (see e.g. the monograph [20] and the recent contributions [16, 19]). In this context, a relevant role is played by the Skew Normal distributions. Indeed, multivariate Skew Normal distributions can be looked upon as a generalization of the Gaussian distributions. One of their key features is the introduction of an additional shape parameter governing the asymmetry of the density function. In so doing, Skew Normal distributions are able to capture several aspects of applied science, meaning that they are suitable for a wide range of application, including finance and management science. They were initially introduced by Azzalini and Dalla Valle [5] in 1996. After their seminal contribution, such distributions have been extensively studied and analyzed in a large number of papers. Just to cite a few, a list of relevant statistical applications is provided in Azzalini and Capitanio [4] in 1999, whereas in 2001 Branco and Dey [12] extend Azzalini and Dalla Valle's methods to multivariate Skew Elliptical distributions. Moreover, a further class of multivariate Skew Normal distributions is introduced by Gupta *et al.* [15] in 2004. In the next year, Azzalini ([3], 2005) presents a rich overview on the family of Skew Normal of distributions and on their generalizations for continuous random variables sharing the same generating mechanism.

Among the most recent contributions, it is worth mentioning at least two articles where Skew Normal distributions are used to deal with real data: an application to HIV–RNA by Ghosh *et al.* [14] in 2007 and a methodology for measurement error based on scale mixtures of Skew Normal distributions. There is also a further stream of studies on finite mixtures of multivariate Skew Normal distributions (e.g. Cabral *et al.* [13], 2012 and Lee and Mclachlan [18], 2013).

To the best of our knowledge, Skew Normal distributions have been studied in detail in the univariate case. As far as the multivariate framework is concerned, scanty attention has been paid to an explicit formulation of the expectation of such random variables conditioned to the fact that a prefixed barrier is not ever crossed. Such a conditional expectation, known as tail conditional expectation, has been calculated in Bernardi ([8], 2013) for univariate Skew Normal distributions and their mixtures. The conditional expectation is a highly relevant concept in financial

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applications: namely, it is quite frequent that expectation of a random variable must be calculated based on some given conditioning event. For example, the conditioning event might be something occurring in the financial market or in the bank system, such as the distress of an important institution, possibly leading to the diffusion of distress among the remaining institutions. In Bernardi *et al.* ([6], 2016) the contagion risk in a financial framework is taken into account, to evaluate the systemic relevance of interconnected institutions. For this purpose, the instrument constructed by Bernardi *et al.* is called SCoVaR and it turns out to be an extension of CoVaR, designed by Adrian and Brunnermeier [2] in 2016. Bernardi *et al.* ([7], 2017), instead makes explicit use of the the Skew Normal generating mechanism to calculate the asymptotic distribution of the Network CoVaR, a statistical procedure to test the pairwise systemic dominance of a financial institution over another one. Further insights on the relation between risk assessment and Skew Normal distributions can be found in ([8] 2013) and in ([10] 2017).

Namely, in the absence of specific formulas for conditional expectation of random variables distributed according to the Skew Normal law, the present paper intends to fill this gap. In particular, we aim at providing an explicit formula for the bivariate Tail Conditional Expectation (TCE, hereafter), defined as

$$\text{TCE}^{\leq} = \mathsf{E}\left(X \mid X \leq \overline{X}\right), \tag{1}$$

where $X = (X_1, X_2)$ is a bivariate Skew Normal random variable and $\overline{X} = \left(\overline{X}_1, \overline{X}_2\right) \in \mathbb{R}^2$. Clearly, $\overline{X}$ can be interpreted as a suitable benchmark, based on the chosen application.

When dealing with bivariate Skew Normal distributions it is worth noting that the bivariate TCE defined in equation (1) strongly differs from the expectation of the involved conditional random variable either from a probabilistic and a risk management perspective. From the probabilistic point of view, it is undoubtedly true that a special distribution of the skew family, namely, the Extended Skew Normal distribution, is closed under marginalization and conditionalization. Therefore an alternative formulation of the tail conditional expectation in equation (1) can be calculated with reference to the univariate conditional distribution of the bivariate Skew Normal, see Bernardi ([8], 2013). However, such a conditional expected value, i.e., $\text{TCE}^{=} = \mathsf{E}\left(X_1 \leq \overline{X}_1 \mid X_2 = \overline{X}_2\right)$ does not involve the bivariate distribution of the random variable (X_1, X_2). From the risk management perspective, instead, the employed definition of tail conditional expectation is not without consequences. Indeed, as recently noted by Bernardi *et al.* ([10], 2017), the $\text{TCE}^{=}$ risk measure suffers the lack of the consistency property since it does not preserve the stochastic ordering induced by the bivariate distribution. This is especially true for high values of the correlation between variables. The $\text{TCE}^{\leq}$ defined in equation (1) instead preserves the stochastic ordering of the bivariate distribution and it can be effectively used as measure of risk.

Several reasonings motivate the relevance of the provided formula for the bivariate TCE under the Skew Normal assumption. First, it generalises the concept of bivariate TCE to a non–Gaussian framework being characterised by the presence of asymmetry, while collapsing to the usual Gaussian case when the involved random variables are symmetric. Second, the TCE provides a convenient way to calculate the CoES risk measure (see, e.g., Adrian and Brunnermeier [2], 2016 and Bernardi and Catania [9], 2015) by simply plugging–in the marginal Value–at–Risk levels. More importantly, as far as it concerns the theoretical properties, the TCE satisfies the sub–additive axiom thereby being a coherent risk measure while the corresponding quantile–based measure (the extension of the VaR, namely, the CoVaR) is not necessarily coherent, see Acerbi and Tasche ([1], 2002) for a comprehensive

discussion of the axiomatic theory of risk measures. The coherence property is even more relevant when mixtures of Skew distributions are involved. The provided formula for TCE easily extends to finite mixture models by adopting the approach of Bernardi ([8], 2013) and Bernardi *et al.* ([11], 2017). As the final remark, the TCE can be effectively exploited to build a network–based systemic risk measure as in Bernardi *et al.* ([6], 2016).

The remainder of the paper can be outlined as follows: Section 2 introduces the statement and the basic notation of the problem. Section 3 collects the procedure and the main results for the explicit formulation of the conditional expectation and Section 4 concludes.

2. The bivariate Skew Normal distribution

We are going to carry out the above procedure to determine the tail conditional expectation when the distribution is Skew Normal (see Azzalini [3], 2005, Azzalini and Dalla Valle [5], 1996, and Gupta *et al.* [15], 2004, among other contributions). For notational convenience, throughout this Section we suppress the symbol ($\leq$) when referring to the TCE as defined in equation (1). In this case, the probability density function is more complex. Namely, beginning from the standard Gaussian unimodal probability density function

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$$

the expression of the probability density function in the bivariate case is

$$f_{SN}(x_1, x_2) = e^{-\frac{1}{2(1-\rho^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} \right]}$$
$$\times \frac{\Phi[\delta_0(x_1 - \mu_1)]\Phi[\gamma_1(x_2 - \mu_2)]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2} \left[\frac{1}{2} - \frac{1}{2\pi} \arccos\left(\frac{\delta_0\gamma_1\sigma_1\sigma_2\rho}{\sqrt{(1+\delta_0^2\sigma_1^2)(1+\gamma_1^2)}} \right) \right]}, \quad (2)$$

where σ_1, σ_2, ρ have the standard meanings, whereas δ_0 and γ_1 are asymmetric parameters. For an extended explanation, see Gupta *et al.* ([15], 2004). In order to lighten the notation, call

$$K = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2} \left[\frac{1}{2} - \frac{1}{2\pi} \arccos\left(\frac{\delta_0\gamma_1\sigma_1\sigma_2\rho}{\sqrt{(1 + \delta_0^2\sigma_1^2)(1 + \gamma_1^2)}} \right) \right],$$

the normalising constant of the previous equation (2). The TCE of X_1 is (see Gupta *et al.* [15], 2004) is defined as:

$$\text{TCE}_{X_1} = \mathsf{E}[X_1 \mid X \leq \overline{X}] = \frac{\mathsf{E}[X_1 \mathbb{1}(X \leq \overline{X})]}{\mathsf{P}(X \leq \overline{X})}, \quad (3)$$

where $\mathbb{1}(\cdot)$ denotes the indicator variable, i.e., $\mathbb{1}(x \in A) = 1$ if and only if $x \in A$ and zero otherwise, while $\mathsf{P}(X \leq \overline{X})$ is the joint probability that the bivariate random variable $X = (X_1, X_2)$ falls below the predetermined threshold $\overline{X} = (\overline{X}_1, \overline{X}_2)$. The denominator of equation (3) can be easily calculated as the cumulative distribution function of the bivariate Skew Normal distribution in equation (2), as follows

$$\mathsf{P}(X \leq \overline{X}) = \int_{-\infty}^{\overline{X}_2} \int_{-\infty}^{\overline{X}_1} f_{SN}(x_1, x_2) \, dx_1 dx_2.$$

As concerns the numerator of equation (3), we have

$$\mathsf{E}[X_1 \mathbb{1}\left(X \leq \overline{X}\right)] = \int_{-\infty}^{\overline{X}_2} \int_{-\infty}^{\overline{X}_1} x_1 f_{SN}(x_1, x_2) dx_2 dx_1$$

$$= \frac{1}{K} \int_{-\infty}^{\overline{X}_1} x_1 \int_{-\infty}^{\overline{X}_2} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]}$$

$$\times \Phi(\gamma_1(x_2 - \mu_2)) dx_2 \Phi(\delta_0(x_1 - \mu_1)) dx_1. \tag{4}$$

We can exploit a typical change of variables, i.e., $t_j = \dfrac{x_j - \mu_j}{\sigma_j}$, for $j = 1, 2$. The determinant of the related Jacobian matrix is $\dfrac{1}{\sigma_1\sigma_2}$, which leads to the following integral

$$\mathsf{E}[X_1 \mathbb{1}\left(X \leq \overline{X}\right)] = \frac{1}{\sigma_1\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} (\sigma_1 t_1 + \mu_1) \Phi(\delta_0 \sigma_1 t_1)$$

$$\times \left[\int_{-\infty}^{\frac{\overline{X}_2-\mu_2}{\sigma_2}} \Phi(\gamma_1 \sigma_2 t_2) \mathrm{e}^{-\frac{1}{2(1-\rho^2)}[t_1^2 + t_2^2 - 2\rho t_1 t_2]} dt_2\right] dt_1$$

$$= J_1 + J_2, \tag{5}$$

where

$$J_1 = \frac{1}{\sigma_1\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} \sigma_1 t_1 \Phi(\delta_0 \sigma_1 t_1)$$

$$\times \left[\int_{-\infty}^{\frac{\overline{X}_2-\mu_2}{\sigma_2}} \Phi(\gamma_1 \sigma_2 t_2) \mathrm{e}^{-\frac{1}{2(1-\rho^2)}[t_1^2 + t_2^2 - 2\rho t_1 t_2]} dt_2\right] dt_1$$

$$= \frac{1}{\sigma_1\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} \sigma_1 t_1 \Phi(\delta_0 \sigma_1 t_1) J_3 dt_1 \tag{6}$$

$$J_2 = \frac{1}{\sigma_1\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} \mu_1 \Phi(\delta_0 \sigma_1 t_1)$$

$$\times \left[\int_{-\infty}^{\frac{\overline{X}_2-\mu_2}{\sigma_2}} \Phi(\gamma_1 \sigma_2 t_2) \mathrm{e}^{-\frac{1}{2(1-\rho^2)}[t_1^2 + t_2^2 - 2\rho t_1 t_2]} dt_2\right] dt_1$$

$$= \frac{1}{\sigma_1\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} \mu_1 \Phi(\delta_0 \sigma_1 t_1) J_3 dt_1, \tag{7}$$

with $J_3 = \int_{-\infty}^{\frac{\overline{X}_2-\mu_2}{\sigma_2}} \Phi(\gamma_1 \sigma_2 t_2) \mathrm{e}^{-\frac{1}{2(1-\rho^2)}[t_1^2 + t_2^2 - 2\rho t_1 t_2]} dt_2$ which is a function of t_1. Hereafter, the dependence of the quantities denoted by J_k on the random variables has been suppressed for notational convenience. Hence, the TCE we intend to calculate explicitly is decomposed into the sum of the two integrals J_1 and J_2. In the next Section we are going to describe the approach to compute them.

3. Calculation of the TCE

The complete calculation of the TCE (4) must be carried out in successive steps. Our procedure can be outlined as follows:

   (i) since both integrals (6) and (7) contain the same integral, which will be denoted by J_3, we focus on the explicit calculation of J_3;

(ii) since J_3 can be further decomposed into the sum of 2 integrals, say J_4 and J_5, we separately consider them; we recognize that J_4 can be calculated, whereas J_5 contains an integral of the kind $\int y^k e^{-y^2} dy$, where k is an integer number;

(iii) the 2 occurrences regarding J_5 have to be treated separately: if k is odd, the integral can be computed; if k is even, J_5 contains an integral of the kind $J_6 = \int_{-\infty}^{a} \Phi(t)dt$, where $\Phi(t)$ is the cumulative distribution function of the standard unimodal Gaussian distribution;

(iv) J_6 can be approximated by series expansion, consequently the expression of J_3 is accomplished;

(v) finally, J_1 and J_2 can be calculated by replacing the above formulation of J_3; in this form, some coefficients and integrals appear, but they can be determined by comparison with the original formulations of J_1 and J_2.

As can be easily inferred, this procedure needs to be separated in several steps, due to the complications which will occur during the computation. The starting point is the integral in square brackets appearing in both (6) and (7). Call it J_3, defined as

$$J_3 = \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} \Phi(\gamma_1 \sigma_2 t_2) e^{-\frac{t_1^2 + t_2^2 - 2\rho t_1 t_2}{2(1-\rho^2)}} dt_2. \tag{8}$$

A simple transformation leads to

$$J_3 = e^{-\frac{t_1^2}{2}} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} \Phi(\gamma_1 \sigma_2 t_2) e^{-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}} dt_2. \tag{9}$$

The integral in (9) can be approximated by using a series expansion of the Gaussian function Φ. Standard mathematical analysis theory establishes that

$$\Phi(\gamma_1 \sigma_2 t_2) = \frac{1}{2} + \frac{1}{\pi} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n (\gamma_1 \sigma_2 t_2)^{2n+1}}{n!(2n+1)} \right]. \tag{10}$$

Plugging (10) into (9) yields:

$$J_3 = e^{-\frac{t_1^2}{2}} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} e^{-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}} \left\{ \frac{1}{2} + \frac{1}{\pi} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n (\gamma_1 \sigma_2 t_2)^{2n+1}}{n!(2n+1)} \right] \right\} dt_2$$

$$= e^{-\frac{t_1^2}{2}} \left[\frac{1}{2} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} e^{-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}} dt_2 \right.$$

$$+ \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n (\gamma_1 \sigma_2)^{2n+1}}{n!(2n+1)} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} t_2^{2n+1} e^{-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}} dt_2 \right]$$

$$= e^{-\frac{t_1^2}{2}} \left[\frac{1}{2} J_4 + \frac{1}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n (\gamma_1 \sigma_2)^{2n+1}}{n!(2n+1)} J_5 \right], \tag{11}$$

where

$$J_4 = \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} e^{-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}} dt_2 \tag{12}$$

$$J_5 = \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2}} t_2^{2n+1} e^{-\frac{(t_2 - \rho t_1)^2}{2(1-\rho^2)}} dt_2. \tag{13}$$

Integrals J_4 and J_5 appearing in (11) can be treated by 2 different changes of variables. First, consider J_4. By setting

$$z = \frac{t_2 - \rho t_1}{\sqrt{1 - \rho^2}},$$

we have

$$J_4 = \sqrt{1 - \rho^2} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{1 - \rho^2}} - \frac{\rho t_1}{\sqrt{1 - \rho^2}}} e^{-\frac{z^2}{2}} dz$$

$$= \sqrt{1 - \rho^2} \Phi \left(\frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{1 - \rho^2}} - \frac{\rho t_1}{\sqrt{1 - \rho^2}} \right). \tag{14}$$

On the other hand, J_5 needs another change of variable. Call

$$y = \frac{t_2 - \rho t_1}{\sqrt{2(1 - \rho^2)}},$$

consequently we have

$$J_5 = \sqrt{2(1 - \rho)^2} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{2(1 - \rho^2)}} - \frac{\rho t_1}{\sqrt{2(1 - \rho^2)}}} (y\sqrt{2(1 - \rho^2)} + \rho t_1)^{2n+1} e^{-y^2} dy$$

$$= \sqrt{2(1 - \rho)^2} \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{2(1 - \rho^2)}} - \frac{\rho t_1}{\sqrt{2(1 - \rho^2)}}} e^{-y^2}$$

$$\times \left\{ \sum_{k=0}^{2n+1} \binom{2n+1}{k} [y\sqrt{2(1 - \rho^2)}]^k (\rho t_1)^{2n+1-k} \right\} dy$$

$$= \sqrt{2(1 - \rho)^2} \sum_{k=0}^{2n+1} \binom{2n+1}{k} [\sqrt{2(1 - \rho^2)}]^k (\rho t_1)^{2n+1-k}$$

$$\times \int_{-\infty}^{\frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{2(1 - \rho^2)}} - \frac{\rho t_1}{\sqrt{2(1 - \rho^2)}}} y^k e^{-y^2} dy. \tag{15}$$

In order to simplify the notation, call

$$\alpha = \alpha_1 + \alpha_2 t_1 = \frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{2(1 - \rho^2)}} - \frac{\rho t_1}{\sqrt{2(1 - \rho^2)}}. \tag{16}$$

The integral in the right–hand side of (15) can be solved by distinguishing the cases of k even or odd.

(i) k is an odd number. This means that there exists an integer h such that $k = 2h + 1$. By making the change of variable $t = y^2$, we obtain that

$$\int y^{2h+1} e^{-y^2} dy = \frac{1}{2} \int t^h e^{-t} dt.$$

By iterating the integration by parts and replacing the variable t with y^2 at the end of the procedure, we have:

$$\int_{-\infty}^{\alpha} y^{2h+1} e^{-y^2} dy$$

$$= -e^{-y^2}$$

$$\times \left[y^{2h} + hy^{2(h-1)} + h(h-1)y^{2(h-2)} + \cdots + h!(y^2 + 1) \right] \Big|_{-\infty}^{\alpha}$$

$$= -e^{-\alpha^2}$$

$$\times \left[\alpha^{2h} + h\alpha^{2(h-1)} + h(h-1)\alpha^{2(h-2)} + \cdots + h!(\alpha^2 + 1) \right]. \tag{17}$$

(ii) k is an even number. We can state that there exists an integer h such that $k = 2h$. Even in this case integration by parts is necessary. After iterating it, the involved integral becomes:

$$\int_{-\infty}^{\alpha} y^{2h} e^{-y^2} dy$$

$$= -\frac{y^{2h-1} e^{-y^2}}{2} - \frac{2h-1}{2^2} y^{2h-3} e^{-y^2} - \frac{(2h-1)(2h-3)}{2^3} y^{2h-5} e^{-y^2}$$

$$- \cdots - \frac{\sqrt{\pi}(2h-1) \cdot (2h-3) \cdots 5 \cdot 3}{2^h} \Phi\left(\frac{y}{\sqrt{2}}\right) \Big|_{-\infty}^{\alpha}$$

$$- \frac{\sqrt{\pi}(2h-1) \cdot (2h-3) \cdots 5 \cdot 3}{2^h} \int_{-\infty}^{\frac{\alpha}{\sqrt{2}}} \Phi(t) dt. \tag{18}$$

Now, call $J_6 = \int_{-\infty}^{\frac{\alpha}{\sqrt{2}}} \Phi(t) dt$. Standard integration results yield

$$J_6 = \int_{-\infty}^{\frac{\alpha}{\sqrt{2}}} \Phi(t) dt$$

$$= [x\Phi(x) + \phi(x)] \Big|_{-\infty}^{\frac{\alpha}{\sqrt{2}}}$$

$$= \frac{\alpha}{\sqrt{2}} \Phi\left(\frac{\alpha}{\sqrt{2}}\right) + \phi\left(\frac{\alpha}{\sqrt{2}}\right) - \lim_{z \to -\infty} (z\Phi(z) + \phi(z))$$

$$= \frac{\alpha}{\sqrt{2}} \Phi\left(\frac{\alpha}{\sqrt{2}}\right) + \phi\left(\frac{\alpha}{\sqrt{2}}\right), \tag{19}$$

since $\lim_{z \to -\infty} (z\Phi(z) + \phi(z)) = 0$. By plugging (19) into (18) we obtain:

$$\int_{-\infty}^{\alpha} y^{2h} e^{-y^2} dy = -\frac{\alpha^{2h-1} e^{-\alpha^2}}{2} - \frac{2h-1}{2^2} \alpha^{2h-3} e^{-\alpha^2}$$

$$- \frac{(2h-1)(2h-3)}{2^3} \alpha^{2h-5} e^{-\alpha^2}$$

$$- \cdots - \frac{\sqrt{\pi}(2h-1) \cdot (2h-3) \cdots 5 \cdot 3}{2^h} \Phi\left(\frac{\alpha}{\sqrt{2}}\right)$$

$$- \frac{\sqrt{\pi}(2h-1) \cdots (2h-3) \cdots 5 \cdot 3}{2^h}$$

$$\times \left[\frac{\alpha}{\sqrt{2}} \Phi\left(\frac{\alpha}{\sqrt{2}}\right) + \phi\left(\frac{\alpha}{\sqrt{2}}\right)\right]. \tag{20}$$

Now, the expressions (17) (for odd values of k) and (20) (for even values of k) have to be inserted into (15). By distinguishing the odd and even cases, integral J_5 turns

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out to be:

$$
\begin{aligned}
J_5 &= \sqrt{2(1-\rho)^2} \sum_{h=0}^{n} \binom{2n+1}{2h} [\sqrt{2(1-\rho^2)}]^{2h} (\rho t_1)^{2n+1-2h} \int_{-\infty}^{\alpha} y^{2h} e^{-y^2} dy \\
&\quad + \sqrt{2(1-\rho)^2} \sum_{h=0}^{n} \binom{2n+1}{2h+1} [\sqrt{2(1-\rho^2)}]^{2h+1} (\rho t_1)^{2n-2h} \int_{-\infty}^{\alpha} y^{2h+1} e^{-y^2} dy \\
&= \sqrt{2(1-\rho)^2} \sum_{h=0}^{n} \Bigg\{ \binom{2n+1}{2h} [\sqrt{2(1-\rho^2)}]^{2h} (\rho t_1)^{2n+1-2h} \\
&\quad \times \Bigg\{ -\frac{\alpha^{2h-1} e^{-\alpha^2}}{2} - \frac{2h-1}{4} \alpha^{2h-3} e^{-\alpha^2} - \frac{(2h-1)(2h-3)}{2^3} \alpha^{2h-5} e^{-\alpha^2} \\
&\quad - \cdots - \frac{\sqrt{\pi}(2h-1) \cdot (2h-3) \cdots 5 \cdot 3}{2^h} \Phi\left(\frac{\alpha}{\sqrt{2}}\right) \\
&\quad - \frac{\sqrt{\pi}(2h-1) \cdot (2h-3) \cdot 5 \cdot 3}{2^h} \left[\frac{\alpha}{\sqrt{2}} \Phi\left(\frac{\alpha}{\sqrt{2}}\right) + \phi\left(\frac{\alpha}{\sqrt{2}}\right) \right] \Bigg\} \Bigg\} \\
&\quad + \sqrt{2(1-\rho)^2} \sum_{h=0}^{n} \binom{2n+1}{2h+1} [\sqrt{2(1-\rho^2)}]^{2h+1} (\rho t_1)^{2n-2h} \\
&\quad \times \left[-e^{-\alpha^2} \left\{ \alpha^{2h} + h\alpha^{2(h-1)} + h(h-1)\alpha^{2(h-2)} + \cdots + h!(\alpha^2+1) \right\} \right].
\end{aligned}
\tag{21}
$$

Although the approximated expression of J_5 is particularly complex, it can be finally employed to accomplish a suitable form for J_3. Integrals J_4 and J_5 are going to be inserted into J_3, subsequently J_3 will be inserted into J_1.

In other words, (14) and (15) are plugged into (11), which is plugged into (6). The expression of α in (16) must be considered as well. At the end of this tedious

and long process, and for a sufficiently large value of Γ, we obtain:

$$
\begin{aligned}
J_1 = {} & \frac{1}{\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \Phi(\delta_0 \sigma_1 t_1) \left\{ \mathrm{e}^{-\frac{t_1^2}{2}} \frac{\sqrt{1-\rho^2}}{2} \Phi\left(\frac{\overline{X}_2 - \mu_2}{\sigma_2 \sqrt{1-\rho^2}} - \frac{\rho t_1}{\sqrt{1-\rho^2}} \right) \right. \\
& + \frac{\mathrm{e}^{-\frac{t_1^2}{2}}}{\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n (\gamma_1 \sigma_2)^{2n+1}}{n!(2n+1)} \\
& \times \left(\sqrt{2(1-\rho)^2} \sum_{h=0}^{n} \binom{2n+1}{2h} [\sqrt{2(1-\rho^2)}]^{2h} (\rho t_1)^{2n+1-2h} \right. \\
& \qquad \times \left\{ -\frac{1}{2} (\alpha_1 + \alpha_2 t_1)^{2h-1} \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} \right. \\
& \qquad\quad - \frac{2h-1}{2^2} (\alpha_1 + \alpha_2 t_1)^{2h-3} \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} \\
& \qquad\quad - \frac{(2h-1)(2h-3)}{2^3} (\alpha_1 + \alpha_2 t_1)^{2h-5} \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} \\
& \qquad\quad - \cdots - \frac{\sqrt{\pi}(2h-1)\cdot(2h-3)\cdots 5 \cdot 3}{2^h} \Phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) \\
& \qquad\quad - \frac{\sqrt{\pi}(2h-1)\cdot(2h-3)\cdots 5 \cdot 3}{2^h} \left[\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \Phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) \right. \\
& \qquad\quad \left.\left. + \phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) \right] \right\} \\
& \qquad + \sqrt{2(1-\rho)^2} \sum_{h=0}^{n} \binom{2n+1}{2h+1} [\sqrt{2(1-\rho^2)}]^{2h+1} (\rho t_1)^{2n-2h} \\
& \qquad\quad \times \left[-\mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} \left\{ (\alpha_1 + \alpha_2 t_1)^{2h} + h(\alpha_1 + \alpha_2 t_1)^{2(h-1)} \right.\right. \\
& \qquad\quad \left.\left.\left. + h(h-1)(\alpha_1 + \alpha_2 t_1)^{2(h-2)} + \cdots + h![(\alpha_1 + \alpha_2 t_1)^2 + 1] \right\} \right] \right) \right\} dt_1,
\end{aligned}
\tag{22}
$$

which reduces to

$$
\begin{aligned}
J_1 = {} & \frac{1}{\sigma_2 K} \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \Phi(\alpha_3 t_1) \left\{ \mathrm{e}^{-\frac{t_1^2}{2}} \frac{\sqrt{1-\rho^2}}{2} \Phi\left(\alpha_4 + \alpha_5 t_1 \right) \right. \\
& + \frac{\mathrm{e}^{-\frac{t_1^2}{2}}}{\pi} \sum_{n=0}^{+\infty} \alpha_6^{(n)} \left(\sum_{h=0}^{n} \alpha_7^{(n,h)} t_1^{2n+1-2h} \left\{ \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} \sum_{s=1}^{h-1} \alpha_8^{(h)} (\alpha_1 + \alpha_2 t_1)^{\alpha_9^{(h)}} \right.\right. \\
& \quad + \alpha_9 \Phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) \\
& \quad \left.\left. - \left[(\alpha_{10} + \alpha_{11} t_1) \Phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) + \phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) \right] \right\} \right. \\
& \quad \left.\left. + \sum_{h=0}^{n} \alpha_{12}^{(n)} t_1^{2n-2h} \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} \sum_{s=0}^{h} \alpha_{13}^{(h,s)} (\alpha_1 + \alpha_2 t_1)^{2h-2s} \right) \right\} dt_1,
\end{aligned}
\tag{23}
$$

where the α's are constants that can be determined by comparing the two terms of the equality in (23). By rearranging terms, (23) is equal to:

$$
\begin{aligned}
J_1 = \frac{1}{\sigma_2 K} \Bigg\{ & \frac{\sqrt{1-\rho^2}}{2} \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \Phi(\alpha_3 t_1) \mathrm{e}^{-\frac{t_1^2}{2}} \Phi\left(\alpha_4 + \alpha_5 t_1\right) dt_1 \\
& + \frac{1}{\pi} \sum_{n=0}^{+\infty} \alpha_6^{(n)} \Bigg[\sum_{h=0}^{n} \alpha_7^{(n,h)} \Bigg(\sum_{s=1}^{h-1} \alpha_8^{(h)} \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} \Phi(\alpha_3 t_1) t_1^{2n+2-2h} \\
& \qquad\qquad\qquad \times \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2 - \frac{t_1^2}{2}} (\alpha_1 + \alpha_2 t_1)^{2h-2s+1} dt_1 \\
& + \alpha_9 \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} \mathrm{e}^{-\frac{t_1^2}{2}} t_1^{2n+2-2h} \Phi(\alpha_3 t_1) \Phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) dt_1 \\
& - \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} \mathrm{e}^{-\frac{t_1^2}{2}} t_1^{2n+2-2h} \Phi(\alpha_3 t_1) (\alpha_{10} + \alpha_{11} t_1) \Phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) dt_1 \\
& + \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} \mathrm{e}^{-\frac{t_1^2}{2}} t_1^{2n+2-2h} \Phi(\alpha_3 t_1) \phi\left(\frac{\alpha_1 + \alpha_2 t_1}{\sqrt{2}} \right) dt_1 \\
& + \sum_{h=0}^{n} \alpha_{12}^{(n)} \sum_{s=0}^{h} \alpha_{13}^{(h,s)} \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \mathrm{e}^{-\frac{t_1^2}{2}} \Phi(\alpha_3 t_1) t_1^{2n-2h} \\
& \qquad\qquad\qquad \times \mathrm{e}^{-(\alpha_1 + \alpha_2 t_1)^2} (\alpha_1 + \alpha_2 t_1)^{2h-2s} dt_1 \Bigg] \Bigg\} \\
= \frac{1}{\sigma_2 K} \Bigg\{ & \frac{\sqrt{1-\rho^2}}{2} \cdot J_7 + \frac{1}{\pi} \sum_{n=0}^{+\infty} \alpha_6^{(n)} \Bigg[\sum_{h=0}^{n} \alpha_7^{(n,h)} \Bigg(\sum_{s=1}^{h-1} \alpha_8^{(h)} \cdot J_8 + \\
& + \alpha_9 \cdot J_9 - J_{10} + J_{11} \Bigg) + \sum_{h=0}^{n} \alpha_{12}^{(n)} \sum_{s=0}^{h} \alpha_{13}^{(h,s)} \cdot J_{12} \Bigg] \Bigg\}, \qquad (24)
\end{aligned}
$$

where the definition of the integrals J_k can be easily derived by comparing the left- and right-hand sides of equation (24). We treat separately integrals J_k:

$$
J_7 = \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \Phi(\alpha_3 t_1) \mathrm{e}^{-\frac{t_1^2}{2}} \Phi\left(\alpha_4 + \alpha_5 t_1\right) dt_1.
$$

We can use the expansion of the Gaussian in (10) and then the standard formula for the power of the binomials. After this, we rewrite J_7 as follows:

$$
\begin{aligned}
J_7 = & \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \mathrm{e}^{-\frac{t_1^2}{2}} \left\{ \frac{1}{2} + \frac{1}{\pi} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha_3 t_1)^{2n+1}}{n!(2n+1)} \right] \right\} \\
& \times \left\{ \frac{1}{2} + \frac{1}{\pi} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha_4 + \alpha_5 t_1)^{2n+1}}{n!(2n+1)} \right] \right\} dt_1 \\
= & \int_{-\infty}^{\frac{\overline{X}_1 - \mu_1}{\sigma_1}} t_1 \mathrm{e}^{-\frac{t_1^2}{2}} \left\{ \frac{1}{2} + \frac{1}{\pi} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha_3 t_1)^{2n+1}}{n!(2n+1)} \right] \right\} \\
& \times \left\{ \frac{1}{2} + \frac{1}{\pi} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n \sum_{s=0}^{2n+1} \binom{2n+1}{s} \alpha_4^s (\alpha_5 t_1)^{2n+1-s}}{n!(2n+1)} \right] \right\} dt_1. \qquad (25)
\end{aligned}
$$

The integral in (25) is fully solved when a solution is provided to the following four integrals:

$$J_{7\alpha} = \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} t_1^\alpha e^{-\frac{t_1^2}{2}} dt_1, \qquad \alpha = 1, 2n+2, 2n+2-s, 4n+3-s. \qquad (26)$$

If $\alpha = 1$, integration is elementary, yielding:

$$J_7 = -e^{-\frac{\overline{Z}_1^2}{2}}, \qquad (27)$$

where $\overline{Z}_1 = \left(\frac{\overline{X}_1-\mu_1}{\sigma_1}\right)$. The cases $\alpha = 2n+2, 2n+2-s, 4n+3-s$ have been already treated above. Note that $\alpha = 2n+2$ is even whereas $\alpha = 2n+2-s$ ($\alpha = 4n+3-s$) is even when s is even (s is odd) and odd otherwise. When α is even, the integral $J_{7\alpha}$ can be approximated as in (20). When α is odd, then the integral $J_{7\alpha}$ is as in (17). Also J_8 can be properly treated by using the series expansion of the Gaussian in (10) and the binomial expansion. Recall that

$$J_8 = \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} \Phi(\alpha_3 t_1) t_1^{2n+2-2h} e^{-(\alpha_1+\alpha_2 t_1)^2 - \frac{t_1^2}{2}} (\alpha_1+\alpha_2 t_1)^{2h-2s+1} dt_1$$

We have

$$J_8 = \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} t_1^{2n+2-2h} e^{-\frac{t_1^2}{2}-(\alpha_1+\alpha_2 t_1)^2} \left\{ \frac{1}{2} + \frac{1}{\pi}\left[\sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha_3 t_1)^{2n+1}}{n!(2n+1)}\right] \right\}$$
$$\times \left[\sum_{r=0}^{2h-2s+1} \binom{2h-2s+1}{r} \alpha_1^r (\alpha_2 t_1)^{2h-2s+1-r}\right] dt_1. \qquad (28)$$

Consider the exponential function in (28). The exponent can be rewritten as follows:

$$-\frac{t_1^2}{2} - (\alpha_1+\alpha_2 t_1)^2 = -\left[t_1\sqrt{\frac{1}{2}+\alpha_2^2} + \frac{\alpha_1\alpha_2}{\sqrt{\frac{1}{2}+\alpha_2^2}}\right]^2 - \alpha_1^2\left(1 - \frac{\alpha_2^2}{\frac{1}{2}+\alpha_2^2}\right). \qquad (29)$$

By substituting (29) into (28) one gets:

$$J_8 = e^{-\alpha_1^2\left(1-\frac{\alpha_2^2}{\frac{1}{2}+\alpha_2^2}\right)} \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} t_1^{2n+2-2h} e^{-\left[t_1\sqrt{\frac{1}{2}+\alpha_2^2}+\frac{\alpha_1\alpha_2}{\sqrt{\frac{1}{2}+\alpha_2^2}}\right]^2}$$
$$\times \left(\frac{1}{2} + \frac{1}{\pi}\left[\sum_{n=0}^{+\infty}\frac{(-1)^n(\alpha_3 t_1)^{2n+1}}{n!(2n+1)}\right]\right)$$
$$\times \left[\sum_{r=0}^{2h-2s+1}\binom{2h-2s+1}{r}\alpha_1^r(\alpha_2 t_1)^{2h-2s+1-r}\right] dt_1. \qquad (30)$$

Now an intuitive change of variable can be made:

$$u = t_1\sqrt{\frac{1}{2}+\alpha_2^2} + \frac{\alpha_1\alpha_2}{\sqrt{\frac{1}{2}+\alpha_2^2}}, \qquad (31)$$

which entails

$$t_1 = \frac{u}{\sqrt{\frac{1}{2}+\alpha_2^2}} - \frac{\alpha_1\alpha_2}{\frac{1}{2}+\alpha_2^2}. \qquad (32)$$

By substituting (31) and (32) into (30) we obtain:

$$
J_8 = \frac{e^{-\alpha_1^2\left(1-\frac{\alpha_2^2}{\frac{1}{2}+\alpha_2^2}\right)}}{\sqrt{\frac{1}{2}+\alpha_2^2}} \int_{-\infty}^{\frac{(\overline{X}_1-\mu_1)\sqrt{1/2+\alpha_2^2}}{\sigma_1}+\frac{\alpha_1\alpha_2}{1/2+\alpha_2^2}} \left[\frac{u}{\sqrt{\frac{1}{2}+\alpha_2^2}}-\frac{\alpha_1\alpha_2}{\frac{1}{2}+\alpha_2^2}\right]^{2n+2-2h}
$$

$$
\times\, e^{-u^2}\left\{\frac{1}{2}+\frac{1}{\pi}\left[\sum_{n=0}^{+\infty}\frac{(-1)^n\left(\alpha_3\left[\frac{u}{\sqrt{\frac{1}{2}+\alpha_2^2}}-\frac{\alpha_1\alpha_2}{\frac{1}{2}+\alpha_2^2}\right]\right)^{2n+1}}{n!(2n+1)}\right]\right\}
$$

$$
\times\left[\sum_{r=0}^{2h-2s+1}\binom{2h-2s+1}{r}\alpha_1^r\left(\alpha_2\left[\frac{u}{\sqrt{\frac{1}{2}+\alpha_2^2}}-\frac{\alpha_1\alpha_2}{\frac{1}{2}+\alpha_2^2}\right]\right)^{2h-2s+1-r}\right]\,du.
$$
(33)

Hence, integral J_8 is analogous to J_7, therefore it can be solved through the same strategy.

Integrals $J_9,\ldots,J_{12}$ can be computed in a similar way as well. In particular, the adoptions of the expansion of the Gaussian and of the binomial powers lead to the fact that J_9 and J_{10} can be directly treated analogously to J_7. Contrary to that, the computation of J_{11} and J_{12} first requires a suitable change of variable (31), and so its procedure is akin to the one adopted for studying J_8. We are going to omit the tedious details[1].

The argument developed above can be adopted in a straightforward way to compute the integral J_2, and J_2 can be written by adapting formula (24). The integral J_2 can be approximated by taking a sufficiently high value of $K > 0$. Namely, we have:

$$
J_2 = \frac{1}{\sigma_2 K}\left\{\frac{\sqrt{1-\rho^2}}{2}\cdot\tilde{J}_7 + \frac{1}{\pi}\sum_{n=0}^{+\infty}\alpha_6^{(n)}\left[\sum_{h=0}^{n}\alpha_7^{(n,h)}\left(\sum_{s=1}^{h-1}\alpha_8^{(h)}\cdot\tilde{J}_8\right.\right.\right.
$$

$$
\left.\left.\left.+\alpha_9\cdot\tilde{J}_9-\tilde{J}_{10}+\tilde{J}_{11}\right)+\sum_{h=0}^{n}\alpha_{12}^{(n)}\sum_{s=0}^{h}\alpha_{13}^{(h,s)}\cdot\tilde{J}_{12}\right]\right\},
$$
(34)

where $\tilde{J}_m$ can be obtained by J_m by multiplying the integrand by the factor t_1^{-1}, for each $m = 7, 8, \ldots, 12$. The integrals to be solved for J_2 belong to the same family of those already treated for J_1. The only new integral occurs in the formulation of $\tilde{J}_7$, which now includes also

$$
\tilde{J}_{70} = \int_{-\infty}^{\frac{\overline{X}_1-\mu_1}{\sigma_1}} e^{-\frac{t_1^2}{2}}\,dt_1 = \Phi\left(\frac{\overline{X}_1-\mu_1}{\sigma_1}\right)\sqrt{2\pi}.
$$

4. Conclusions and further developments

This paper is entirely devoted to the complex calculation of the TCE of a random vector which is composed of bivariate Skew Normal random variables. We outlined the whole procedure and developed most steps in an extended form. From the risk management perspective it is worth noting that the provided risk measure can be effectively used to assess the systemic risk contributions of different institutions belonging to a given market extending the SCoVaR measure introduced by Bernardi *et al.* ([6], 2016). Furthermore, the provided risk measure enjoys the consistency property since it preserves the stochastic ordering induced by the bivariate distribution.

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[1]However, all calculations are available for the interested readers upon request to the authors.

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