Another Map from $\mathbb{P}^7$ to the Study Quadric

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Abstract

In [5] Pfurner, Schröcker and Husty introduced a mapping from $\mathbb{P}^7$ to the Study quadric. In [9], it was shown that this map could be thought of as the composition of an extended version of the inverse Cayley map based on the 6×6 adjoint representation of the group, and the Cayley map itself. Here, the analogous map using the Cayley map based on the standard 4×4 representation of $SE(3)$ is studied. It is shown that mapping a general line in $\mathbb{P}^7$ results in a motion with cubic trajectories. A different view of the map is then studied. A birational map between the Study quadric and the variety defined by the adjoint representation of the group is given. The new map is then the composition of the map from the Study quadric, extended to all $\mathbb{P}^7$, with the map from the $\mathbb{P}^{17}$ back to the Study quadric. The effect of the new map on symmetric subspaces of $SE(3)$ is also considered. Lastly, an example is given showing how the interpolation techniques can be extended to point constraints. That is, where a point on the body is required to pass through a sequence of successive points in space.

1 Introduction

Pfurner, Schröcker and Husty,[5], introduced an explicit rational map from the projective space $\mathbb{P}^7$ to the Study quadric Q_s. The Study quadric is a non-singular quadric hypersurface in $\mathbb{P}^7$. Points of this quadric (with the exception of those in a single generator 3-plane) are in 1-to-1 correspondence with the elements of the group of proper isometries of $\mathbb{R}^3$—the group of rigid-body displacements in space; $SE(3)$. Motivating this work was the wish to produce a simple method to interpolate rigid-body motions. From knot point in the group a rational interpolation is constructed and then mapped to Q_s to give a rational rigid-body motion. In [9], it was shown that this map is closely connected to the Cayley map associated to the adjoint representation of $SE(3)$. The inverse of the Cayley map extends to all $\mathbb{P}^7$ then Pfurner, Schröcker and Husty's map (PSH map) is the composition of this extended map, with the Cayley map from the Lie algebra of the group to the group itself.

Here another such map is introduced. This map is based on the Cayley map of the 4×4 standard representation of $SE(3)$. Generally, the new map produces motions of lower degree than the PSH map. In particular the map of a general line in $\mathbb{P}^7$ is conic in the Study quadric. This conic represents a rigid-body motion with cubic trajectories, a motion previously studied by Wunderlich [10].

A different view of the PSH map is given in [6]. There the map is thought of as a composition of maps between the Study quadric and the variety determined by the standard 4×4 homogeneous representation of $SE(3)$. The map from the Study quadric extends simply to a map from all $\mathbb{P}^7$ to the representation of the group, and hence the composition maps from $\mathbb{P}^7$ to the Study quadric. Here, it is shown that the new map can be viewed in a similar way but using the adjoint representation of the group. To do this the maps between the Study quadric and the variety determined by the representation are found.

In [9] it was noted that the algebraic symmetric subspaces of $SE(3)$ are either linear subspaces in Q_s or the intersection of Q_s with some linear subspace of $\mathbb{P}^7$. In [9] it was also noted that the PSH map preserves these linear subspaces. So the PSH map can be used to interpolate motions in symmetric subspaces of the group without change. Here, the effect of the new map is also considered.

Finally an example is given showing how the interpolation technique can be extended to interpolation of point data. In some situations a point on the body is required to pass through successive points in space, the orientation of the body being irrelevant at these knot points. In the Study quadric the set of all displacements which move a particular point to another given point form a 3-plane generator of the quadric. Extra information is required to find a unique group element on this 3-plane.

First some notation will be set up.

2 Maps from $\mathbb{P}^7$ to the Study Quadric

A general dual quaternion is given by,

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k) \tag{1}$$

where i, j and k are the unit quaternion generators and ε is the dual unit which commutes with the quaternions and squares to zero, $\varepsilon^2 = 0$.

A rigid-body displacement is given by a dual quaternion with elements satisfying the equation,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0. \tag{2}$$

Taking $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ as real homogeneous coordinates in a $\mathbb{P}^7$, the above quadratic equation determines the Study quadric Q_S. Elements of this projective quadric (with the exception of the points on the 3-plane A_∞ determined by $a_0 = a_1 = a_2 = a_3 = 0$), are in 1-to-1 correspondence with elements of the group of rigid-body displacements, $SE(3)$.

2.1 The PSH Map

The map given in [5] takes an arbitrary point of $\mathbb{P}^7$ to Q_S in $\mathbb{P}^7$, so let us write the coordinates in the first $\mathbb{P}^7$ as $\bar{a}_i$ and $\bar{c}_i$, now the map can be written as,

$$\begin{aligned}
a_0 &= \bar{a}_0(\bar{a}\cdot\bar{a}), & c_0 &= \bar{c}_0(\bar{a}\cdot\bar{a}) - \bar{a}_0(\bar{a}\cdot\bar{c}),\\
a_1 &= \bar{a}_1(\bar{a}\cdot\bar{a}), & c_1 &= \bar{c}_1(\bar{a}\cdot\bar{a}) - \bar{a}_1(\bar{a}\cdot\bar{c}),\\
a_2 &= \bar{a}_2(\bar{a}\cdot\bar{a}), & c_2 &= \bar{c}_2(\bar{a}\cdot\bar{a}) - \bar{a}_2(\bar{a}\cdot\bar{c}),\\
a_3 &= \bar{a}_3(\bar{a}\cdot\bar{a}), & c_3 &= \bar{c}_3(\bar{a}\cdot\bar{a}) - \bar{a}_3(\bar{a}\cdot\bar{c}).
\end{aligned} \tag{3}$$

where $\bar{a}\cdot\bar{a} = \bar{a}_0^2 + \bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_3^2$ and $\bar{a}\cdot\bar{c} = \bar{a}_0\bar{c}_0 + \bar{a}_1\bar{c}_1 + \bar{a}_2\bar{c}_2 + \bar{a}_3\bar{c}_3$. It is straightforward to check that the image of this map satisfies the equation defining the Study quadric and hence the image of the map is indeed Q_S.

2.2 The New Map

Taking the homogeneous coordinates of $\mathbb{P}^7$ as $(\bar{a}_0 : \bar{a}_1 : \bar{a}_2 : \bar{a}_3 : \bar{c}_0 : \bar{c}_1 : \bar{c}_2 : \bar{c}_3)$ and the coordinates of the Study quadric as $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ the new map can be written explicitly as,

$$\begin{array}{rclrcl} a_0 & = & \bar{a}_0^2, & c_0 & = & -(\bar{a}_1\bar{c}_1 + \bar{a}_2\bar{c}_2 + \bar{a}_3\bar{c}_3), \\ a_1 & = & \bar{a}_0\bar{a}_1, & c_1 & = & \bar{a}_0\bar{c}_1, \\ a_2 & = & \bar{a}_0\bar{a}_2, & c_2 & = & \bar{a}_0\bar{c}_2, \\ a_3 & = & \bar{a}_0\bar{a}_3, & c_3 & = & \bar{a}_0\bar{c}_3. \end{array} \tag{4}$$

Clearly this map takes points in $\mathbb{P}^7$ to points on the Study quadric. Notice that it is only quadratic in the homogeneous coordinates of $\mathbb{P}^7$ while the PSH map (3) is cubic.

The map has an exceptional set, that is a set of points in $\mathbb{P}^7$ not mapped to any point. This consists of the intersection of the Study quadric Q_s, with its tangent hyperplane at the point ε, that is the point $(0:0:0:0:1:0:0:0)$. The hyperplane has the equation $\bar{a}_0 = 0$ and hence the intersection with Q_s is a quadric cone in this $\mathbb{P}^6$ with equation $\bar{a}_1\bar{c}_1 + \bar{a}_2\bar{c}_2 + \bar{a}_3\bar{c}_3 = 0$. The vertex of the cone is the point ε. This cone can also be described as the set of π-screws, that is rotations of π radians about lines in space composed with an arbitrary translations along the line.

Fibres of the map are lines through the point ε. Suppose that g is a point in Q_s then the line $g + \lambda\varepsilon$ clearly maps to g. Note also that points on the tangent hyperplane to ε, away from the exceptional set, are mapped to ε.

2.3 The 4×4 Cayley map

In [7] Cayley maps were found for both the standard 4×4 representation of $SE(3)$ and the 6×6 adjoint representation of the group. In [9] it was shown that the PSH map can be thought of as the inverse 6×6 Cayley map extended to all of $\mathbb{P}^7$ composed with the Cayley map back to the Study quadric. Here the analogous compositions using the 4×4 Cayley map will be studied.

Both of these maps can be turned into birational maps between the Study quadric and $\mathbb{P}^6$. That is, the Cayley map and its inverse can be viewed as a birational transformation between the six-dimensional projective space $\mathbb{P}^6$ and the Study quadric in $\mathbb{P}^7$. To see this, introduce a homogenising variable w_0 so the homogeneous coordinates for the $\mathbb{P}^6$ will be $(w_0 : w_1 : w_2 : w_3 : u_1 : u_2 : u_3)$ and the coordinates in the $\mathbb{P}^7$ will be $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ as above.

The Cayley map is a map from twists to group elements. If a general twist is written as the dual quaternion,

$$s = (w_1 i + w_2 j + w_3 k) + \varepsilon(u_1 i + u_2 j + u_3 k),$$

then the 4×4 Cayley map can be written as a polynomial,

$$\mathrm{Cay}_4(s) = \frac{1}{2\sqrt{w_0^2 + |w|^2}}\big((2w_0^2 + |w|^2) + 2w_0 s + s^2\big),$$

where the variable w_0 has been incorporated to render the equation homogeneous. Explicitly this is,

$$\begin{array}{rclrcl} a_0 & = & w_0^2, & c_0 & = & -(w_1u_1 + w_2u_2 + w_3u_3), \\ a_1 & = & w_0w_1, & c_1 & = & w_0u_1, \\ a_2 & = & w_0w_2, & c_2 & = & w_0u_2, \\ a_3 & = & w_0w_3, & c_3 & = & w_0u_3. \end{array}$$

Note that, the normalising factor, $1/2\sqrt{w_0^2 + |w|^2}$, can be cancelled since the coordinates are homogeneous. So the Cayley map is a quadratic transformation.

The inverse map, Cay_4^{-1} is given by a cubic polynomial,

$$\begin{aligned}\mathrm{Cay}_4^{-1}(g) &= \frac{1}{2a_0(a_0^2+a_1^2+a_2^2+a_3^2)}\Big(g^3 - 4a_0 g^2 + \\ &\qquad (4a_0^2 + 3(a_0^2+a_1^2+a_2^2+a_3^2))g - 4a_0(a_0^2+a_1^2+a_2^2+a_3^2)\Big).\end{aligned}$$

If we assign $w_0 = 2a_0(a_0^2+a_1^2+a_2^2+a_3^2)$, the common denominator, then the other coordinates of $s = \mathrm{Cay}_4^{-1}(g)$ are given by expanding the polynomial in g and simplifying using the relation $a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0$ defining the Study quadric. After cancelling the common factor, $2(a_0^2 + a_1^2 + a_2^2 + a_3^2)$, the result is,

$$\begin{aligned} w_0 &= a_0 & & \\ w_1 &= a_1 & u_1 &= c_1 \\ w_2 &= a_2 & u_2 &= c_2 \\ w_3 &= a_3 & u_3 &= c_3 \end{aligned}$$

So this is, in fact, a linear projection, and is clearly the inverse to the Cayley map above. The centre of this linear projection is just the point $(0 : 0 : 0 : 0 : 1 : 0 : 0 : 0)$ which, as a dual quaternion is simply ε. This simple projection clearly extends to the whole of $\mathbb{P}^7$. The extended map will be written, $\widetilde{\mathrm{Cay}_4}^{-1}$.

From this it is clear that the new map, described above, is the composition $\mathrm{Cay}_4 \circ \widetilde{\mathrm{Cay}_4}^{-1}$.

2.4 The Image of a General Line

Other lines in $\mathbb{P}^7$ map to conics in the Study quadric. These lines meet Q_s in two points, together with ε this forms a plane meeting Q_s in a conic curve. This curve through ε must be the image of the line.

For example, suppose the line meets the Study quadric at the identity, the point 1, and the screw displacement,

$$g = (c + sk) + \varepsilon(-\frac{p\theta}{2}s + \frac{p\theta}{2}ck)$$

where $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$ are constants. This displacement has pitch p and its screw axis is the z-axis. The line joining these points will have coordinates,

$$\begin{aligned} \bar{a}_0 &= \lambda + \mu c, & \bar{c}_0 &= -\mu\tfrac{p\theta}{2}s, \\ \bar{a}_1 &= 0, & c_1 &= 0, \\ \bar{a}_2 &= 0, & \bar{c}_2 &= 0, \\ \bar{a}_3 &= \mu s, & \bar{c}_3 &= \mu\tfrac{p\theta}{2}c, \end{aligned}$$

where λ and μ are the homogeneous parameters of the line. The image of this line under the new map is then,

$$\begin{aligned} a_0 &= (\lambda + \mu c)^2, & c_0 &= -\mu^2\tfrac{p\theta}{2}cs, \\ a_1 &= 0, & c_1 &= 0, \\ a_2 &= 0, & c_2 &= 0, \\ a_3 &= (\lambda + \mu c)\mu s, & c_3 &= (\lambda + \mu c)\mu\tfrac{p\theta}{2}c. \end{aligned}$$

An affine transformation of the parameters is useful here: let $\alpha = \lambda + \mu c$ and $\beta = \mu s$. With these parameters the conic is given by,

$$g(\alpha : \beta) = (\alpha^2 + \alpha\beta k) + \varepsilon(-\beta^2\kappa + \alpha\beta\kappa k),$$

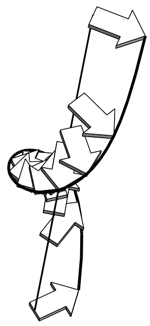

Figure 1: The arrow moves according to a special Wunderlich motion about an axis. The curve is the trajectory of the arrow's point.

where the constant $\kappa = p\theta c/2s$.

Notice that the conic meets A_∞, the 3-plane of non-physical displacements, in a single point at the parameter value $\alpha = 0$. Now, if $1 + \varepsilon(xi + yj + zk)$ is a point in space then we can find the trajectory of the point under the action of the motion from,

$$W + \varepsilon(Xi + Yj + Zk) = g(\alpha : \beta)\Big(w + \varepsilon(xi + yj + zk)\Big)g^\dagger(\alpha : \beta). \tag{5}$$

Here the homogenising variables w and W have been introduced so that the curves are now curves in a 3-dimensional projective space $\mathbb{P}^3$. The superscript †, in the above is intended to denote the conjugation:

$$g^\dagger = (a + \varepsilon c)^\dagger = (a^- - \varepsilon c^-).$$

Substituting the rigid-body motion $g(\alpha : \beta)$ gives results which are of degree 4 in α and β, but in each term α is a common factor and so may be cancelled. The results are cubic trajectories,

$$\begin{array}{rcl} X &=& \alpha(\alpha^2 - \beta^2)x - 2\alpha^2\beta y, \\ Y &=& 2\alpha^2\beta x + \alpha(\alpha^2 - \beta^2)y, \\ Z &=& \alpha(\alpha^2 + \beta^2)z + 2\beta(\alpha^2 + \beta^2)\kappa, \\ W &=& \alpha(\alpha^2 + \beta^2)w. \end{array}$$

So, the motion must be a special type of Wunderlich motion. In fact we may reparameterise the curve again, so that $\alpha = \cos(\phi/2)$ and $\beta = \sin(\phi/2)$, then the curve can be written as an affine curve,

$$\begin{array}{rcl} X &=& x\cos\phi - y\sin\phi, \\ Y &=& x\sin\phi + y\cos\phi, \\ Z &=& z + 2\kappa\tan(\phi/2). \end{array}$$

Note that this result is quite general, any line meeting the Study quadric in two places can be left-translated so that one of the intersection points coincides with the identity, then coordinates can be chosen so that the z-axis of space coincides with the screw axis of the second displacement. So using the new map to interpolate a motion between a pair of displacements will give this special type of cubic motion, see figure 1.

Rigid-body motions where almost all points in space follow cubic trajectories were studied by W. Wunderlich [10]. The general such motion was found to be given by certain quartic curves in the Study quadric. These motions are characterised by 6 parameters. Here all but one of the parameters is zero and the quartic degenerates to a conic, see [2].

2.5 A Different View

In [6] Schröcker takes a different view of these maps. Rather than considering maps from $\mathbb{P}^7$ to the Study quadric, he looks at a map from $\mathbb{P}^7$ to the projective compactification of the standard 4×4 representation of $SE(3)$ in $\mathbb{P}^{12}$. An element of the group of rigid-body displacements is given by a matrix of the form,

$$\begin{pmatrix} R & \vec{t} \\ 0 & \Delta \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & \Delta \end{pmatrix}$$

where the components are homogeneous coordinates in a twelve dimensional projective space. In terms of these variables the map can be written,

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 - a_0a_3) & 2(a_1a_3 + a_0a_2) \\ 2(a_1a_2 + a_0a_3) & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2(a_2a_3 - a_0a_1) \\ 2(a_1a_3 - a_0a_2) & 2(a_2a_3 + a_0a_1) & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}$$

for the rotation matrix,

$$\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 2 \begin{pmatrix} a_0c_1 - a_1c_0 + a_2c_3 - a_3c_2 \\ a_0c_2 - a_1c_3 - a_2c_0 + a_3c_1 \\ a_0c_3 + a_1c_2 - a_2c_1 - a_3c_0 \end{pmatrix}$$

for the translation vector and finally the homogenising variable is given by,

$$\Delta = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

This map takes any point of $\mathbb{P}^7$ to the group. That is, whatever the values of a_i and c_i, so long as not all of the a_i are zero, then the image will satisfy $RR^T = \Delta^2 I_3$ and $\det(R) = \Delta^3$, with I_3 the 3×3 identity matrix.

Schröcker also considers the inverse map from $\mathbb{P}^{12}$ to the Study quadric given by,

$$\begin{aligned} a_0 &= \tfrac{1}{4}(\Delta + r_{11} + r_{22} + r_{33})\Delta, \\ a_1 &= \tfrac{1}{4}(r_{32} - r_{23})\Delta, \\ a_2 &= \tfrac{1}{4}(r_{13} - r_{31})\Delta, \\ a_3 &= \tfrac{1}{4}(r_{21} - r_{12})\Delta, \\ \\ c_0 &= \tfrac{1}{8}\big((r_{23} - r_{32})t_1 + (r_{31} - r_{13})t_2 + (r_{12} - r_{21})t_3\big), \\ c_1 &= \tfrac{1}{8}\big((\Delta + r_{11} + r_{22} + r_{33})t_1 - (r_{12} - r_{21})t_2 + (r_{31} - r_{13})t_3\big), \\ c_2 &= \tfrac{1}{8}\big((r_{12} - r_{21})t_1 + (\Delta + r_{11} + r_{22} + r_{33})t_2 - (r_{23} - r_{32})t_3\big), \\ c_3 &= \tfrac{1}{8}\big((r_{13} - r_{31})t_1 + (r_{23} - r_{32})t_2 + (\Delta + r_{11} + r_{22} + r_{33})t_3\big). \end{aligned}$$

Actually, Schröcker considers four different maps and linear combinations of the four. This particular map is chosen here since it maps the identity matrix to the dual quaternion 1. Now when these two maps are composed to produce a map from $\mathbb{P}^7$ to the Study quadric the result is the PSH map described above. The image of a line in $\mathbb{P}^{12}$ in the Study quadric is again a special Wunderlich motion even though the construction here is not the same as that described in section 2.4.

Notice that, these maps between $\mathbb{P}^7$ and $\mathbb{P}^{12}$ were also given in [8]. Also given in this previous work was a rational map from the Study quadric to the 6×6 adjoint representation of $SE(3)$.

In the adjoint representation of $SE(3)$ group elements are given by matrices of the form,

$$\mathrm{Ad}(g) = \begin{pmatrix} M & 0 \\ U & M \end{pmatrix}$$

where M and U are 3×3 matrices. Taking the elements of these two matrices as homogeneous coordinates we can see that the adjoint representation lies in a $\mathbb{P}^{17}$. The map from the Study quadric is given by,

$$\begin{aligned} M &= \Delta I_3 + 2a_0 A + 2A^2, \\ U &= 2(a_0 C + c_0 A + AC + CA), \end{aligned}$$

here, as before, a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3 are the coordinates in $\mathbb{P}^7$. Again as before, I_3 is the 3×3 identity matrix and $\Delta = a_0^2 + a_1^2 + a_2^2 + a_3^2$. The 3×3 matrices A and C are anti-symmetric matrices,

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}.$$

It is simple to see that this map extends to all of $\mathbb{P}^7$ and that the image of the map lies in the group. To show that the image of the map lies in $SE(3)$ we need to verify that M is a multiple of a rotation matrix and that U is the product of an anti-symmetric matrix with M. Simple but lengthy computations, best suited to a computer algebra system, shows that $MM^T = \Delta^2 I_3$, $\det(M) = \Delta^3$ and $UM^T + MU^T = 0$.

The inverse of this map was not found in [8]. Note that, since this is a map between projective spaces, multiplying the coordinates by a non-zero common factor is irrelevant. Moreover, since this is only a rational map the constant can be zero as long as it is not zero everywhere. In order to present the map from $\mathbb{P}^{17}$ to the Study quadric in a reasonably readable form it is necessary to introduce some intermediate variables. So let,

$$\overline{M} = (M - M^T) \qquad \text{and} \qquad \overline{U} = (U - U^T).$$

The map is then given by,

$$\begin{aligned} a_0 &= \big(12\det(M) + 4\,\mathrm{Tr}(M)\,\mathrm{Tr}(MM^T)\big)\big(\mathrm{Tr}(MM^T) - \mathrm{Tr}(M^2)\big) \\ A &= 4\overline{M}\big(\mathrm{Tr}(MM^T) - \mathrm{Tr}(M^2)\big)\,\mathrm{Tr}(MM^T) \\[1ex] c_0 &= 2\big(\mathrm{Tr}(MM^T) - \mathrm{Tr}(M^2)\big)\,\mathrm{Tr}(MM^T)\,\mathrm{Tr}(U) \\ C &= -4\big(\overline{M}^2\,\overline{U} - 2\overline{M}\,\overline{U}\,\overline{M} + \overline{U}\,\overline{M}^2\big)\,\mathrm{Tr}(MM^T) \\ &\quad - \overline{M}\big(2\,\mathrm{Tr}(M^3) + 2\,\mathrm{Tr}(MM^T)\,\mathrm{Tr}(M) - 3\,\mathrm{Tr}(M^2)\,\mathrm{Tr}(M) + \mathrm{Tr}(M)^3\big)\,\mathrm{Tr}(U) \end{aligned}$$

It is clear that this map extends to a map from all of $\mathbb{P}^{17}$ and is of degree 5 in the coordinates of $\mathbb{P}^{17}$. It can be verified that the image of this map satisfies equation (2), so the image of the map lies in the Study quadric.

Finally here, composing the two maps; the one from $\mathbb{P}^7$ to $\mathbb{P}^{17}$ with the map from $\mathbb{P}^{17}$ to the Study quadric gives a map from $\mathbb{P}^7$ to the Study quadric. Up to multiplication by an overall factor of $768\,a_0^2\Delta^2$ this is the new map given in equation (4).

3 Groups and Symmetric Spaces

In [3] Nawratil introduced an interpretation of the ambient space $\mathbb{P}^7$, containing the Study quadric, as a subgroup of $SE(4)$, the rigid-body motions in 4 dimension. This seven dimensional subgroup of $SE(4)$ is an analogue of the Schönflies group in 3-dimensions. In 3D a Schönflies subgroup preserves the orientation of a fixed 2-plane. In 4D the subgroup X_4, is the group of rigid-body displacements that maintain the orientation of a given 3-plane. It is not too difficult to see that the subgroup is isomorphic to the semi-direct product $X_4 = SO(3) \rtimes \mathbb{R}^4 \subset SE(4)$. That is the semi-direct product of the rotations preserving the 3-plane with the 4-dimensional abelian subgroup of pure translations. Note that in 4 dimensions the axis of a rotation is a 2-plane. Further, since the action of the 3-dimensional rotation group fixes translations in one particular direction, that is elements of $SO(3)$ commute with translations in one direction, it can be seen that the subgroup is also isomorphic to the direct product, $X_4 = SE(3) \times \mathbb{R} \subset SE(4)$.

In terms of dual quaternions, points in $\mathbb{R}^4$ can be written as, $1 + \varepsilon(x_0 + x_1 i + x_2 j + x_3 k)$ where $(x_0,\, x_1\; x_2,\, x_3)$ are the coordinates of the point. Now all elements of $SE(3)$ can be written as dual quaternions of the form,

$$(a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k) = g + \frac{1}{2}\varepsilon\vec{t}g,$$

where g is a non-zero quaternion, that is $gg^- \neq 0$ and $\vec{t}$ is a pure quaternion, so that $\vec{t}^- = -\vec{t}$. The quaternion g, represents the rotational part of the displacement and the pure quaternion $\vec{t}$, gives the translational part. It is not difficult to see that dual quaternions of this form lie on the Study quadric since the equation for the Study quadric, (2), can be written,

$$ac^- + ca^- = 0,$$

where $a = a_0 + a_1 i + a_2 j + a_3 k$ and similarly $c = c_0 + c_1 i + c_2 j + c_3 k$. Substituting $a = g$ and $c = \frac{1}{2}\vec{t}g$ gives,

$$g(\frac{1}{2}\vec{t}g)^- + \frac{1}{2}\vec{t}gg^- = -\frac{1}{2}gg^-\vec{t} + \frac{1}{2}\vec{t}gg^- = 0,$$

since gg^- is real and so commutes with all quaternions.

The effect of such a displacement on a point in $\mathbb{R}^4$ can be found by extending the action given in equation (5) above,

$$\begin{aligned}(g + \frac{1}{2}\varepsilon\vec{t}g)\big(1 + \varepsilon(x_0 + x_1 i + x_2 j + x_3 k)\big)(g - \frac{1}{2}\varepsilon\vec{t}g)^- \\ = gg^-\Big(1 + \varepsilon\big(\frac{1}{gg^-}g(x_1 i + x_2 j + x_3 k)g^- + x_0 + \vec{t}\big)\Big)\end{aligned}$$

Notice that, the rotation does not effect coordinates in the x_0 direction.

For an arbitrary dual quaternion we can assume that c is the product of two arbitrary quaternions, $c = \frac{1}{2}(t_0 + \vec{t})g$, where $\vec{t}$ and g are as before and t_0 is a real constant. Using this to compute the displacement of a point in $\mathbb{R}^4$ gives,

$$\begin{aligned}(g + \frac{1}{2}\varepsilon(t_0 + \vec{t})g)\big(1 + \varepsilon(x_0 + x_1 i + x_2 j + x_3 k)\big)(g + \frac{1}{2}\varepsilon(t_0 - \vec{t})g)^- \\ = gg^-\Big(1 + \varepsilon\big(\frac{1}{gg^-}g(x_1 i + x_2 j + x_3 k)g^- + x_0 + t_0 + \vec{t}\big)\Big)\end{aligned}$$

This shows that t_0 gives the translation in the x_0 direction and hence verifies Nawratil's interpretation of $\mathbb{P}^7$. The non-zero factor gg^- can, of course, be cancelled since we are working in a projective space.

Nawratil also considered what happens when a dual quaternion is subject to the PSH map. Let $\bar{a} + \varepsilon\bar{c} = g + \frac{1}{2}\varepsilon(t_0 + \vec{t})g$. By writing $g = g_0 + \vec{g}$, the Gibb's relation can be used to simplify computations. First $\bar{a}_0 = g_0$ and $\bar{a}\cdot\bar{a} = g_0^2 + |\vec{g}|^2$. Then,

$$\bar{c} = \frac{1}{2}(t_0 + \vec{t})(g_0 + \vec{g}) = \frac{1}{2}(t_0 g_0 - \vec{t}\cdot\vec{g} + g_0\vec{t} + t_0\vec{g} + \vec{t}\times\vec{g}).$$

Now $\bar{a}\cdot\bar{c} = (1/2)t_0(g_0^2 + |\vec{t}|^2) = (1/2)t_0(\bar{a}\cdot\bar{a})$. See (3) above. Hence, after cancelling the non-zero common factor $\bar{a}\cdot\bar{a}$, we have that,

$$\mathrm{PSH}\Big(g + \frac{1}{2}\varepsilon(t_0 + \vec{t})g\Big) = g + \frac{1}{2}\varepsilon\vec{t}g,$$

that is, the map simply forgets the translation perpendicular to the $x_1x_2x_3$ 3-plane.

Notice that, the above shows that the group of rigid body displacements in 3D can be viewed as the quotient $X_4/\mathbb{R} = SE(3)$. This means that the PSH map is the projection map for the principal bundle, $PSH : X_4 \longrightarrow SE(3)$ with fibre $\mathbb{R}$. Moreover the PSH map is a homomorphism $PSH : X_4 \longrightarrow SE(3)$.

As discussed at the end of section 2.2, the fibres of the new map are lines through ε. Hence, writing the new map as $N()$, it is clear that,

$$\mathrm{N}\Big(g + \frac{1}{2}\varepsilon(\gamma_0 + \vec{t}g)\Big) = g + \frac{1}{2}\varepsilon\vec{t}g,$$

where γ_0 is arbitrary. It is straightforward to show that this is not a homomorphism.

3.1 Symmetric Subspaces of $SE(3)$

In [9] it was shown that algebraic subgroups of $SE(3)$ and algebraic sub-symmetric spaces of $SE(3)$ lie on linear spaces in $\mathbb{P}^7$. The spaces are either linear spaces contained in Q_s or the intersection of Q_s with a linear space. For convenience, Tables 1 and 2 from [9] are reproduced here. These give canonical forms for the subgroups and subsymmetric spaces of $SE(3)$.

Table 1: Canonical Forms for the Connected Subgroups of $SE(3)$. GH type denotes the class of the screw system in the Gibson-Hunt classification of screw systems.

Dim	GH type	Subgroup	Sub. Alg. basis	Linear equations	Description
1	IA ($p = 0$)	$SO(2)$	$\{i\}$	$a_2 = a_3 = c_0 = c_1 = c_2 = c_3 = 0$	line in Q_S
1	IA ($p \neq 0$)	H_p	$\{i + p\varepsilon i\}$	not algebraic	-
1	IIB	$\mathbb{R}$	$\{\varepsilon i\}$	$a_1 = a_2 = a_3 = c_0 = c_2 = c_3 = 0$	line in Q_S
2	IB^0	$SO(2)\times\mathbb{R}$	$\{i, \varepsilon i\}$	$a_2 = a_3 = c_2 = c_3 = 0$	3-plane
2	IIC	$\mathbb{R}^2$	$\{\varepsilon i, \varepsilon j\}$	$a_1 = a_2 = a_3 = c_0 = c_1 = 0$	2-plane in Q_S
3	IIA ($p = 0$)	$SO(3)$	$\{i, j, k\}$	$c_0 = c_1 = c_2 = c_3 = 0$	A-plane
3	IIC ($p = 0$)	$SE(2)$	$\{i, \varepsilon j, \varepsilon k\}$	$a_2 = a_3 = c_0 = c_1 = 0$	A-plane
3	IIC ($p \neq 0$)	$H_p \ltimes \mathbb{R}^2$	$\{i + p\varepsilon i, j, k\}$	not algebraic	-
3	IID	$\mathbb{R}^3$	$\{\varepsilon i, \varepsilon j, \varepsilon k\}$	$a_1 = a_2 = a_3 = c_0 = 0$	B-plane
4	$\overline{\text{IIC}}$	$SE(2)\times\mathbb{R}$	$\{i, \varepsilon i, \varepsilon j, \varepsilon k\}$	$a_2 = a_3 = 0$	5-plane

These linear spaces were shown to be preserved by the PSH map. This is also true for the new map. For example, the Schönfliess subgroup lies in the intersection of Q_s with a 5-plane. The

Table 2: Canonical Forms for the Connected Symmetric Subspaces of $SE(3)$. LTS basis denotes a basis for the Lie triple system.

Dim	GH type	LTS basis	Linear equations	Description
2	IIA ($p = 0$)	$\{i, j\}$	$a_3 = c_0 = c_1 = c_2 = c_3 = 0$	2-plane in Q_S
2	IIB ($p = 0$)	$\{i, \varepsilon j\}$	$a_2 = a_3 = c_0 = c_1 = c_3 = 0$	2-plane in Q_S
2	IIB ($p \neq 0$)	$\{i + p\varepsilon i, \varepsilon j\}$	not algebraic	-
3	IIB ($p = 0$)	$\{i, j, \varepsilon k\}$	$a_3 = c_0 = c_1 = c_2 = 0$	B-plane
3	IC^0	$\{i, \varepsilon i, \varepsilon j\}$	$a_2 = a_3 = c_3 = 0$	4-plane
4	$\overline{\mathrm{IB}}^0$	$\{i, j, \varepsilon i, \varepsilon j\}$	$a_3 = c_3 = 0$	5-plane
5	$\overline{\mathrm{IIB}}$	$\{i, j, \varepsilon i, \varepsilon j, \varepsilon k\}$	$a_3 = 0$	hyperplane

5-plane can be the one defined by $a_2 = a_3 = 0$. Under the new map any point with $\bar{a}_2 = \bar{a}_3 = 0$ will be mapped to one with,

$$a_2 = \bar{a}_0\bar{a}_2 = 0 \quad \text{and} \quad a_3 = \bar{a}_0\bar{a}_3 = 0$$

So for example, if we take a general twist from a IIB ($p = 0$) 3-system, $s = ai + bj + c\varepsilon k$ the exponential of this is,

$$e^s = \cos\theta + \frac{a}{\theta}\sin\theta i + \frac{b}{\theta}\sin\theta j + \frac{c}{\theta}\sin\theta\varepsilon k \tag{6}$$

where $\theta^2 = a^2 + b^2$. Clearly, whatever the values of the parameters a, b and c, the exponential lies in the 3-plane $a_3 = c_0 = c_1 = c_2 = 0$. This 3-plane is a generator plane of the Study quadric.

In this way, all possible subalgebras and Lie triple systems can be examined. Tables of canonical forms for the possible subalgebras and Lie triple systems can be found in Table 1 and 2 respectively, together with the linear equations satisfied by the subspaces they generate.

4 Interpolation through Points

A common problem in the design of mechanisms is to find a rigid-body motion that moves a body in such a way that a point on the body coincides successively with a sequence of prescribed points in space. The orientation of the body when the point on the body coincides with the target points is not relevant. There are many works in the literature addressing this "body-guidance problem", for example [4]. Here this problem of finding a rational rigid-body motion satisfying a number of such point constraints is considered from the viewpoint of the Study quadric.

The set of displacements that move a point $\vec{\alpha}$ to a point $\vec{\beta}$ can be found as follows. First translate $\vec{\alpha}$ back to the origin, perform an arbitrary rotation $a = a_0 + a_1 i + a_2 j + a_3 k$ about the origin, finally translate the origin to $\vec{\beta}$. The dual quaternions q representing this composition of displacements is,

$$q = (1 + \frac{1}{2}\varepsilon\vec{\beta})a(1 - \frac{1}{2}\varepsilon\vec{\alpha}) = a + \frac{1}{2}\varepsilon(\vec{\beta}a - a\vec{\alpha})$$

Hence, $q = a + \varepsilon c$ with $c = \frac{1}{2}(\vec{\beta}a - a\vec{\alpha})$. This shows that c can be expressed linearly in terms of

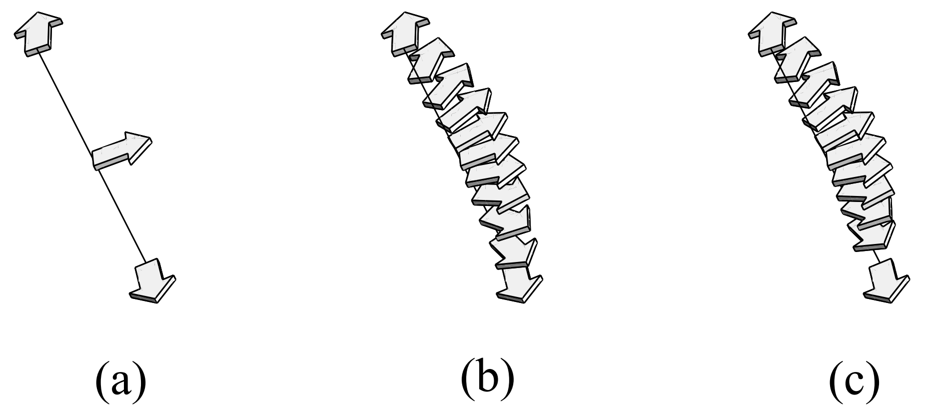

Figure 2: (a) The three points for the interpolation are given by the points of the arrow, the orientations are determined by the intersection of the point constraints with a particular 4-plane. (b) The motion given by mapping the interpolation conic using the PSH map. (c) The same interpolated curve but mapped to a motion using the new map.

the rotation a,

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & (\alpha_x - \beta_x) & (\alpha_y - \beta_y) & (\alpha_z - \beta_z) \\ (\beta_x - \alpha_x) & 0 & -(\alpha_z + \beta_z) & (\alpha_y + \beta_y) \\ (\beta_y - \alpha_y) & (\alpha_z + \beta_z) & 0 & -(\alpha_x + \beta_x) \\ (\beta_z - \alpha_z) & -(\alpha_y + \beta_y) & (\alpha_x + \beta_x) & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{7}$$

This matrix equation represents 4 linear equations which determine an A-plane in the Study quadric. Note, it is straightforward to see that this 3-plane lies in the Q_s since the coefficient matrix is anti-symmetric. In $\mathbb{P}^7$ a 4-plane will intersect a general 3-plane, such as an A-plane, in a unique point. Given several point constraints we can choose a 4-plane in $\mathbb{P}^7$ that meets each A-plane given by a point constraint at a single point. A polynomial curve can be constructed to pass through all of these points. Finally the interpolating curve can be mapped back to Q_s using either the PSH map or the map introduced above.

4.1 An Example

As 4-plane, suppose we choose the plane defined by the three equations $a_2 = a_3 = c_3 = 0$ which intersects the Study quadric in a 3-dimensional subsymmetric space, see table 2. In this case it is possible to combine these three equations with the four equations for the A-plane given in equation (7) and produce a symbolic solution for an arbitrary point constraint:

$$\begin{aligned} q = {} & 2(\alpha_y + \beta_y) + 2(\beta_z - \alpha_z)i \\ & + \varepsilon\big((\alpha_x - \beta_x)(\beta_z - \alpha_z) + (\beta_x - \alpha_x)(\alpha_y + \beta_y)i + (\beta_y^2 + \beta_z^2 - \alpha_y^2 - \alpha_z^2)j\big). \end{aligned}$$

So, q is the dual quaternion, lying on the 4-plane, which moves $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)^T$ to $\vec{\beta} = (\beta_x, \beta_y, \beta_z)^T$.

Suppose we try to approximate a finite screw motion see figure 2 (a), the knot points for the interpolation are the points of the arrows. Using the result above gives three dual quaternions on the Study quadric. The interpolating curve is a conic in $\mathbb{P}^7$. Figure 2 (b) shows the mapping of the interpolating curve using the PSH map while Figure 2 (c) illustrates the result using the new map presented in this work. Note that, the result of using the PSH map is a motion that has point trajectories of degree 6 whilst the point trajectories of the new map have degree 4. On the other hand, from the diagram it can be seen that the motion given by the PSH map has a more uniform speed over the interpolating interval than the new map. Problems can arise with either map, if the interpolating curve in $\mathbb{P}^7$ meets the exceptional set of the map.

4.2 Hybrid Constraints

An advantage of this approach is that mixtures of point and pose constraints can be treated in this framework. Here a pose condition is a specification of the position and orientation that the body must achieve at some time. So, in a hybrid constraint problem a point on the body is required to coincide successively with some given points but at some of these points the orientation of the body is also specified.

For up to five pose constraints this can be solved as follows: Five general pose constraints will define a 4-plane in $\mathbb{P}^7$ and hence 5 pose constraints and n general point constraints will determine $n+5$ points in Q_s. As before a polynomial curve can be interpolated using the $n+5$ points and then mapped to Q_s. With fewer than five pose constraints there will be some freedom to the choice of $\mathbb{P}^4$. More than five pose constraints cannot, in general, be interpolated in this way.

5 Conclusion

An obvious question that arises from this work is: Are there any other maps from $\mathbb{P}^7$ to the Study quadric? In general, Cayley maps for different representations of a group are different. However, for some inequivalent representations the Cayley maps might be the same. For the group $SE(3)$ many inequivalent representations are known but only two Cayley maps. It would be useful to know whether or not there were any others.

As pointed out in [9], the interpolation of poses in $SE(3)$, either using the PSH map or the new map are very similar in conception to the methods outlined in [1]. Here $SE(3)$ is embedded in the general affine group $GA^+(3)$, this is just the semi-direct product of $GL^+(3)$, the positive determinant 3×3 matrices with $\mathbb{R}^3$, the translations. The projection, that maps to $SE(3)$, is essentially given by the polar decomposition of the 3×3 matrix; ignoring the symmetric matrix in the decomposition and retaining only the orthogonal part. It is not too difficult to see that this idea could be extended to any split extension of $SE(3)$ as long as the extension group has a linear structure so that interpolation can be performed in the extended group.

Finally it may be possible to use these ideas for location problems. In many areas of robotics, mobile robots, robot vision and others, a fundamental problem is to find the location of the robot given data from sensors. The sensors may give data on the position of points, lines, planes and sometimes coordinate frames. The problem is to find the position and orientation of the robot from the measured data. If the sensor data is perfect, these location problems are usually fairly simple to solve. But real data will contain errors and so the real problem here is to produce an accurate result given noisy data. The difficulty is that there are usually consistency conditions that data must satisfy. For example, consider trying to find the rigid-body displacement undergone by the end-effector of a robot by measuring the positions of points on the end-effector before and after the displacement. The pairwise distances between the points should be the same before and after the displacement since these distances should be preserved by a rigid-body displacement. Sensor noise and rounding errors will mean that the distances will, almost certainly, not be the exactly the same before and after the displacement. A possible approach is to set-up the problem as a number of linear equations in the dual quaternions. Then we can solve the overdetermined system of equations using pseudo-inverses or similar techniques. Now the solution will be a dual quaternion, but not necessarily a point on the Study quadric, that is not necessarily a rigid-body displacement. So, finally an element of $SE(3)$ can be recovered by mapping the point to the Study quadric.

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References

[1] C. Belta and V. Kumar, 2002, "An SVD-projection method for interpolation on SE(3)", *IEEE Trans. Robotics and Automation*, vol. 18, pp. 334—345.

[2] M. Hamann, 2011, "Line-symmetric motions with respect to reguli", *Mechanism and Machine Theory*, vol. 46, no. 7, pp. 960–974.

[3] G. Nawratil, 2018, "Kinematic Interpretation of the Study Quadric's Ambient Space", in *Advances in Robot Kinematics 2018* Eds. J. Lenarčič and V. Parenti-Castelli, Springer Proceedings in Advanced Robotics vol. 8, pp.3–11, Springer Nature, Cham.

[4] A. Perez and J.M. McCarthy, 2003,"Dimensional synthesis of Bennett linkages", *Journal of Mechanical Design*, vol. 125, no. 1, pp. 98–104.

[5] M. Pfurner, H-P. Schröcker and M. Husty, 2018, "Path Planning in Kinematic Image Space Without the Study Condition", in *Advances in Robot Kinematics 2016*, J. Lenarčič and J-P. Merlet eds. Springer Proceedings in Advanced Robotics vol 4, pp. 27–30, Springer Nature, Cham.

[6] H-P. Schröcker, 2018, "From A to B: New Methods to Interpolate Two Poses". *Journal for Geometry and Graphics*, vol. 22 no. 1 pp. 87–98.

[7] J.M. Selig, 2007, "Cayley maps for $SE(3)$", *The International Fedaration for the Promotion of Mechanism and Machine Science 12th World Congress, Besançon*, Available online at: `http://79.6.136.172/iftomm/proceedings/proceedings\_WorldCongress/WorldCongress07/articles/article\_cd.htm`. A corrected version of this work can be found at, `https://www.researchgate.net/publication/242315171\_Cayley\_Maps\_for\_SE3`

[8] J.M. Selig, 2013, "On the geometry of the homogeneous representation for the group of proper rigid-body displacements", *Romanian Journal of Technical Sciences - Applied Mechanics, (Special issue on "New Trends in Advanced Robotics")* , vol. 58, no. 1–2, pp. 27–50.

[9] J.M. Selig, Yuanquing Wu and M. Carricato, 2018, "Motion Interpolation in Lie Subgroups and Symmetric Subspaces", In S. Zeghloul, L. Romdhane, M. Laribi eds. *Computational Kinematics. Mechanisms and Machine Science*, vol 50, pp. 467–474, Springer Verlag, Cham.

[10] W. Wunderlich, 1984, "Kubische Zwangläufe", *Sitzungsber., Abt. II, Österr, Akad, Wiss., Math, -Naturw* Kl, 193:45—68.