On the Geometry of Some Localisation Problems in Robotics

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Abstract

In this work a couple of localisation problems for mobile robots are revisited. Specifically the problem of finding the location of the robot from the distances to fixed beacons and using time differences of arrivals of signals at several stations are addressed. The aim is to study the geometry of these problems. In particular, tetracyclic coordinates are used to represent circles in the plane. These coordinates are an old idea that have not been used for these types of problem before. The coordinates help to simplify expressions and hence expose the underlying geometric ideas involved in these problems.

Keywords: Localisation, Tetracyclic coordinates, Cyclographic coordinates, Osculating circles.

1 Introduction

In many areas of robotics, mobile robots, robot vision and others, a fundamental problem is to find the location of the robot given data from sensors. The sensors may give data on the position of points, lines, planes and sometimes coordinate frames. The problem is to find the position and orientation of the robot from the measured data.

If the sensor data is perfect, these location problems are often fairly simple to solve. But real data will contain errors and so the real problem here is to produce an accurate result given noisy data. A standard way to improve the accuracy of any measurement is to take more readings and then combining the data using some kind of average. So we should seek methods and algorithms that can use more measurements than are strictly necessary to solve the problem. A probability model for the distribution of errors would help, then we could use Bayesian techniques to find maximum likelihood estimators. But here the idea is to concentrate of the geometry of the situation. A final complication that can arise in these location problems is that often the sensor data must satisfy some consistency conditions for solutions to exist, however, in reality these conditions will rarely be met exactly.

A couple of different location problems in the plane will be examined from the point of view outlined above. Tetracyclic or cyclographic coordinates will be used as an efficient way to express these problems. The focus of the work is on the geometry of the problems not on practical algorithms for solving the problems. The idea is to show that these problems, which are extremely important in robotics, have a significant geometrical content. Indeed there are very close to the original meaning of the word "geometry".

2 Multiateration

Consider the problem of finding the location of a point given its distance to a set of known beacons. In three dimension, with three beacons this problem and its possible solution methods is known as trilateration, with more beacons the term multilateration is sometimes used. The problem arises in many situations such as mobile robotics, surveying and even the determination of molecular structure. Many different technologies are available to measure the distance to the beacon, for example the time-of-flight of ultrasonic pulses can be used or the signal strength of radio frequency signals. Even computer vision techniques can be used. The technology chosen for the task clearly depends on the magnitudes of the distances, the clutter in the environment and other similar considerations. Here we look at the two dimensional version of the problem, though this generalises easily to three dimensions. We begin by looking at representing circles in the plane by points in a projective space $\mathbb{RP}^3$.

2.1 Tetracyclic Coordinates

The equation of a circle in the plane with centre $\vec{c} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$ and radius ρ is,

$$(x - c_x)^2 + (y - c_y)^2 = \rho^2$$

Expanding this equation and rearranging, it can be written as the product of a pair of vectors,

$$\big(-\sqrt{2}x,\ -\sqrt{2}y,\ x^2 + y^2,\ 1 \big) \begin{pmatrix} \sqrt{2}c_x \\ \sqrt{2}c_y \\ 1 \\ c_x^2 + c_y^2 - \rho^2 \end{pmatrix} = 0. \tag{1}$$

Notice that, the row vector on the left contains only quantities associated to a point in the plane while the column vector on the right contains only the parameters of the circle. We can identify the 4-component vectors of the form,

$$\mathring{c} = \begin{pmatrix} \sqrt{2}c_x \\ \sqrt{2}c_y \\ 1 \\ c_x^2 + c_y^2 - \rho^2 \end{pmatrix}$$

with circles in the plane. Clearly every circle in the plane corresponds to such a vector. However, many vectors correspond to the same circle since multiplying the vector by a non-zero constant will not change the circle specified by the equation. So we are led to consider the components of $\mathring{c}$ as homogeneous coordinates for the projective space $\mathbb{PR}^3$. In this space the coordinates $\mathring{c}$ and $\lambda\mathring{c}$ represent the same point, so long as $\lambda \neq 0$. The projective space $\mathbb{PR}^3$ can be thought of as the space of lines through the origin in the 4-dimensional vector space $\mathbb{R}^4$. These coordinates for the circles are called cyclographic coordinates or tetracyclic coordinates, see [1]. A typical point in this $\mathbb{PR}^3$ can be written as $\mathring{k} = (k_1,\, k_2,\, k_3,\, k_4)$ where the k_is are homogeneous coordinates. As stated, this means that the same point is given by, $\mathring{k} = (\lambda k_1,\, \lambda k_2,\, \lambda k_3,\, \lambda k_4)$ when λ is not zero. There are points in $\mathbb{PR}^3$ which don't correspond to real circles,since if we try to find the radius of the circle, using,

$$\frac{1}{2k_3^2}(k_1^2 + k_2^2 - 2k_3k_4) = \rho^2,$$

the result may not be positive. We could get around this by working over the complex numbers and accepting circles with complex radii as valid circles. This approach will not be taken here.

Notice that, this representation of circles contains circles of radius $\rho = 0$, these may be identified with points in the plane. The set of these zero radius circles lie on the zero-set of the degree-2, homogeneous equation,

$$k_1^2 + k_2^2 - 2k_3k_4 = 0, \tag{2}$$

where, as above, the k_is are the homogeneous coordinates of $\mathbb{PR}^3$. This relation can be written in a matrix form as,

$$\mathring{k}^T J \mathring{k} = 0,$$

where,

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Suppose we look at the product, $\mathring{k}^T J \mathring{c}$, where $\mathring{k}$ is a zero-radius circle and $\mathring{c}$ is a circle with non-zero radius, we get,

$$\begin{aligned} &\mathring{k}^T J \mathring{c} = \\ &\qquad \left(\sqrt{2}k_x,\, \sqrt{2}k_y,\, 1,\, k_x^2 + k_y^2\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}c_x \\ \sqrt{2}c_y \\ 1 \\ c_x^2 + c_y^2 - \rho^2 \end{pmatrix} \\ &\qquad\qquad = \rho^2 - (k_x - c_x)^2 - (k_y - c_y)^2. \end{aligned}$$

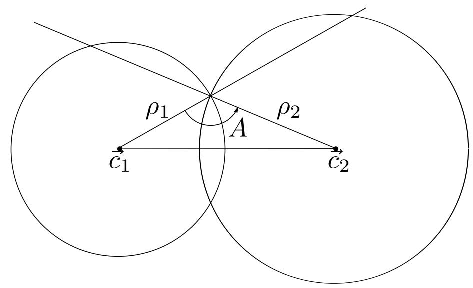


Figure 1: Two Circles Meeting at an Angle.

If this quantity vanishes then the point $(k_x,\, k_y)$ lies on the circle represented by $\mathring{c}$. For a pair of circles $\mathring{c}_1$, $\mathring{c}_2$, with non-zero radii, we have,

$$\mathring{c}_1^T J \mathring{c}_2 =$$

$$\left(\sqrt{2}c_{1x},\, \sqrt{2}c_{1y},\, 1,\, c_{1x}^2 + c_{1y}^2 - \rho_1^2\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}c_{2x} \\ \sqrt{2}c_{2y} \\ 1 \\ c_{2x}^2 + c_{2y}^2 - \rho_2^2 \end{pmatrix}$$

$$= \rho_1^2 + \rho_2^2 - (c_{1x} - c_{2x})^2 - (c_{1y} - c_{2y})^2.$$

If this quantity vanishes then the distance between the centres of the two circles and the two radii of the circles satisfy Pythagoras' theorem. This implies that the two circles meet at two real points, at the points of intersection the tangent lines to the circles are perpendicular. Circles satisfying $\mathring{c}_1^T J \mathring{c}_2 = 0$ are said to be perpendicular circles.

If the quantity, $\mathring{c}_1^T J \mathring{c}_2$ doesn't vanish, then we can use the cosine rule to write,

$$\mathring{c}_1^T J \mathring{c}_2 = 2\rho_1\rho_2 \cos A,$$

where A is the angle between the tangents to the circles at their intersections, see figure 1.

2.2 Three Beacons

Imagine a mobile robot that needs to find its location relative to three beacons. The robot's sensors can measure the distance to the three beacons. The beacons are located at known points,

$$\vec{c}_1 = \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \quad \vec{c}_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \quad \vec{c}_3 = \begin{pmatrix} X_3 \\ Y_3 \end{pmatrix}.$$

Suppose the distances to the robot from the beacons are found to be d_1, d_2, d_3 respectively. Assuming the position of the robot is given by a zero-radius circle

$\mathring{k}$, the sensor data can be expressed as three equations,

$$\mathring{k}^T J \mathring{c}_i = 0, \quad i = 1,\, 2,\, 3. \tag{3}$$

where,

$$\mathring{c}_i = \begin{pmatrix} \sqrt{2}X_i \\ \sqrt{2}Y_i \\ 1 \\ X_1^2 + Y_i^2 - d_i^2 \end{pmatrix}, \qquad i = 1,\, 2,\, 3.$$

A standard solution to this system of equations is found by subtracting one of the equations from the other two, say we subtract the first equation. This removes the terms $k_x^2 + k_y^2$ and produces a pair of equations linear in the unknowns k_x and k_y,

$$\begin{aligned} 0 &= (d_2^2 - d_1^2 - d_2^2 - X_2^2 - X_1^2 - Y_2^2 - Y_1^2) - 2(X_1 - X_2)k_x - 2(Y_1 - Y_2)k_y \\ 0 &= (d_3^2 - d_1^2 - d_3^2 - X_3^2 - X_1^2 - Y_3^2 - Y_1^2) - 2(X_1 - X_3)k_x - 2(Y_1 - Y_3)k_y \end{aligned}$$

These are easily solved to give the position of the robot. The set of linear combinations of two circles in these tetracyclic coordinates give, what is called a pencil of circles. In any such pencil of circles there is always one circle with infinite radius, a line. Geometrically, the above approach replaces the second two circles in the problem specification with the lines in the two pencils of circles formed by the first circle with the second and with the third circles. When the data is exact, this will give the exact solution to the problem. But what if there is some error in the data? Now it will make a difference which equation we subtract from the others. We will get different lines and hence different intersections.

Hence, we seek a more symmetrical approach which does not involve lines. Suppose we treat the three equations above as linear equations in $\mathbb{RP}^3$. Now the solutions can be written as determinants. The three equations can be written in the form,

$$\begin{pmatrix} X_1 & Y_1 & 1 & X_1^2 + Y_1^2 - d_1^2 \\ X_2 & Y_2 & 1 & X_2^2 + Y_2^2 - d_2^2 \\ X_3 & Y_3 & 1 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix} \begin{pmatrix} 2k_x \\ 2k_x \\ -k_x^2 - k_y^2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

The solution is,

$$k_x = \frac{-1}{4A} \det \begin{pmatrix} 1 & Y_1 & X_1^2 + Y_1^2 - d_1^2 \\ 1 & Y_2 & X_2^2 + Y_2^2 - d_2^2 \\ 1 & Y_3 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix} \text{ and } k_y = \frac{1}{4A} \det \begin{pmatrix} 1 & X_1 & X_1^2 + Y_1^2 - d_1^2 \\ 1 & X_2 & X_2^2 + Y_2^2 - d_2^2 \\ 1 & X_3 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix},$$

where

$$A = \frac{1}{2} \det \begin{pmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{pmatrix}$$

is the area of the triangle formed by the beacons, $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$.

Notice that, the original problem just gives the position of the robot in barycentric coordinates based on the three beacons. The solution above is just the conversion between the barycentric coordinates and the Cartesian coordinates in which we have chosen to express the position vectors of the beacons. See [5] for a similar 3-dimensional version.

2.3 Errors

The above formulas produce a result even if there is some error in the measurements, this is because we have chosen to ignore the consistency condition that the solution should be a zero radius circle. When there is some error in the measurements the solution will have a non-zero radius. If we assume that the $\mathring{k}$ in the equations (3) has a non-zero radius then these equations express the fact that the circle $\mathring{k}$ is perpendicular to each of the three circles $\mathring{c}_1$, $\mathring{c}_2$ and $\mathring{c}_3$. Writing,

$$\mathring{k} = \begin{pmatrix} \sqrt{2}X \\ \sqrt{2}Y \\ 1 \\ X^2 + Y^2 - \rho^2 \end{pmatrix}$$

the equations for the circle to be perpendicular to the three circles around the beacons become,

$$\begin{pmatrix} X_1 & Y_1 & 1 & X_1^2 + Y_1^2 - d_1^2 \\ X_2 & Y_2 & 1 & X_2^2 + Y_2^2 - d_2^2 \\ X_3 & Y_3 & 1 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix} \begin{pmatrix} 2X \\ 2Y \\ \rho^2 - X^2 - Y^2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again this can be solved by determinants. The solutions for X and Y are the same as for k_x and k_y respectively. Rearranging the solution gives the radius of the common perpendicular circle as,

$$\begin{aligned} \rho^2 = \frac{1}{16A^2} \Bigg(& \det \begin{pmatrix} 1 & Y_1 & X_1^2 + Y_1^2 - d_1^2 \\ 1 & Y_2 & X_2^2 + Y_2^2 - d_2^2 \\ 1 & Y_3 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix}^2 \\ & + \det \begin{pmatrix} X_1 & 1 & X_1^2 + Y_1^2 - d_1^2 \\ X_2 & 1 & X_2^2 + Y_2^2 - d_2^2 \\ X_3 & 1 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix}^2 \\ & - 8A \det \begin{pmatrix} X_1 & Y_1 & X_1^2 + Y_1^2 - d_1^2 \\ X_2 & Y_2 & X_2^2 + Y_2^2 - d_2^2 \\ X_3 & Y_3 & X_3^2 + Y_3^2 - d_3^2 \end{pmatrix} \Bigg) \end{aligned}$$

This radius might give some indication of the magnitude of the error in the result for the position of the robot. However, it might be better to look for a result that can cope with more measurements.

2.4 Minimum Square of the Residuals

A method suggested in [2] is to minimise the sum of the squares of the residuals given by the equations in (3). The residuals are the quantities r_i given by,

$$\mathring{k}^T J \mathring{c}_i = r_i.$$

The quantity that is to be minimised is given by,

$$\Phi = \frac{1}{n}\sum_{i=1}^{n} r_i^2 = \frac{1}{n}\sum_{i=1}^{n} \mathring{k}^T J \mathring{c}_i \mathring{c}_i^T J \mathring{k} = \mathring{k}^T \mathcal{Q} \mathring{k},$$

where,

$$\mathcal{Q} = \frac{1}{n}\sum_{i=1}^{n} J \mathring{c}_i \mathring{c}_i^T J.$$

Notice that the sum is over n, indicating that the number of beacons that can be used is not limited to 3. The symmetric matrix $\mathcal{Q}$ can be expanded to,

$$\mathcal{Q} = \frac{1}{n}\sum_{i=1}^{n} \begin{pmatrix} 2c_{ix}^2 & 2c_{ix}c_{iy} & -\sqrt{2}c_{ix}\kappa_i & -\sqrt{2}c_{ix} \\ 2c_{ix}c_{iy} & 2c_{iy}^2 & -\sqrt{2}c_{iy}\kappa_i & -\sqrt{2}c_{iy} \\ -\sqrt{2}c_{ix}\kappa_i & -\sqrt{2}c_{iy}\kappa_i & \kappa_i^2 & \kappa_i \\ -\sqrt{2}c_{ix} & -\sqrt{2}c_{iy} & \kappa_i & 1 \end{pmatrix},$$

where $\kappa_i = c_{ix}^2 + c_{iy}^2 - d_i^2$. This is a constrained optimisation problem since we want to constrain the resulting circle to have zero radius. That is, the constraint is given by the quadratic function, $\mathring{k}^T J \mathring{k} = 0$. Now we can differentiate Φ with respect to the coordinates of $\mathring{k}$, include the constraint and set the result to zero. This gives the equation,

$$(\mathcal{Q} + \lambda J)\mathring{k} = 0,$$

where λ is the Lagrange multiplier. For non-trivial solutions we must have,

$$\det(\mathcal{Q} + \lambda J) = 0.$$

This is now a generalised eigenvalue problem. Since $\mathcal{Q}$ and J are 4×4 matrices we can expect up to 4 generalised eigenvalues and hence up to 4 corresponding eigenvectors. The generalised eigenvector which gives the smallest value of Φ is the solution we seek.

If the true distance to the beacon is $\rho_i = d_i + \epsilon_i$ then the residual will be given by, $r_i = \epsilon_i(2d_i + \epsilon_i)$. In the minimisation this means that more weight given to the beacons which are further from the robot, that is the ones with larger values of the distances d_i. This might be appropriate in some circumstances but in general would lead to a biased result.

2.5 Least Squares

A more usual way to deal with errors would be to minimise the sum of the squares of the absolute errors. This approach is not as simple as it might at first

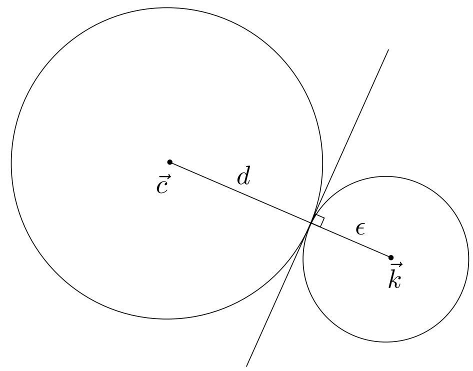


Figure 2: Errors and Osculating Circles.

seem. First we need to introduce another detail of the tetracyclic coordinates. Suppose that $\mathring{k}$ is a circle whose centre is the solution to this problem. As seen at the end of the previous section, the distance between the solution and the i-th beacon is $d_i + \epsilon_i$ where d_i is the measured distance to the beacon and ϵ_i is the error. This can be viewed as a pair of circles meeting with parallel tangents, see figure 2. Circles, or more generally any curves meeting in this fashion are said to osculate or kiss. That is, we can think of $\mathring{k}$ as having radius ϵ_i and kissing the circle $\mathring{c}_i$. The condition for two circles to kiss can be written in terms of tetracyclic coordinates as follows.

Recall from section 2.1, that given a pair of circles $\mathring{k}$ and $\mathring{c}$ we have that,

$$\mathring{k}^T J \mathring{c} = 2\rho_1\rho_2 \cos A,$$

but if the circles are kissing when they meet the angle between them is $A = 0$, or π, hence $\cos A = \pm 1$. Also the radii of the circles are d and ϵ so that,

$$\mathring{k}^T J \mathring{c} = \pm 2d\epsilon.$$

Moreover, we have $\mathring{k}^T J \mathring{k} = 2\epsilon^2$ and $\mathring{c}^T J \mathring{c} = 2d^2$. Combining these results gives,

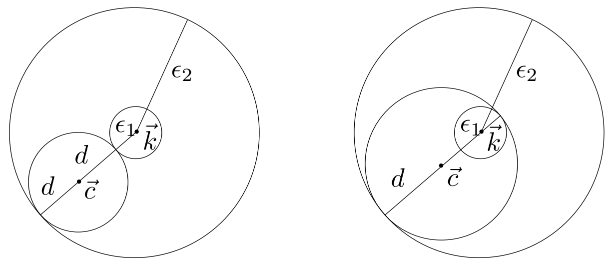
$$(\mathring{k}^T J \mathring{c})^2 - (\mathring{c}^T J \mathring{c})(\mathring{k}^T J \mathring{k}) = 0.$$

This quadratic relation is satisfied if and only if the circles kiss. This relation can be rearranged to,

$$\mathring{k}^T \Big(J\mathring{c}\mathring{c}^T J - (\mathring{c}^T J \mathring{c}) J \Big) \mathring{k} = \mathring{k}^T W \mathring{k} = 0,$$

where W is the 4×4 symmetric matrix, $W = \Big(J\mathring{c}\mathring{c}^T J - (\mathring{c}^T J \mathring{c}) J \Big)$, explicitly this is,

$$W = \begin{pmatrix} 2(c_x^2 - d^2) & 2c_x c_y & -\sqrt{2}c_x\kappa & -\sqrt{2}c_x \\ 2c_x c_y & 2c_y^2 - d^2) & -\sqrt{2}c_y\kappa & -\sqrt{2}c_y \\ -\sqrt{2}c_x\kappa & -\sqrt{2}c_y\kappa & \kappa^2 & \kappa + 2d^2 \\ -\sqrt{2}c_x & -\sqrt{2}y & \kappa + 2d^2 & 1 \end{pmatrix},$$



(a) (b)

Figure 3: Osculating Circles centred on a Fixed Point, (a) Centre Outside, (b) Centre Inside a fixed Circle.

where $\kappa = c_x^2 + c_y^2 - d^2$.

If the position of the robot is fixed then the error can be found from the radius of the circle satisfying, $\mathring{k}^T W \mathring{k} = 0$. Expanding this equation actually gives a quadratic in the square of the error,

$$
\begin{aligned}
0 = \mathring{k}^T W \mathring{k} = \\
&\big((c_x - k_x)^2 + (c_y - k_y)^2 - (d - \epsilon)^2\big)\big((c_x - k_x)^2 + (c_y - k_y)^2 - (d + \epsilon)^2\big).
\end{aligned}
$$

Expanding and gathering terms gives,

$$\epsilon^4 - 2\big((c_x - k_x)^2 + (c_y - k_y)^2 + d^2\big)\epsilon^2 + \big((c_x - k_x)^2 + (c_y - k_y)^2 - d^2\big)^2 = 0.$$

This gives two solutions,

$$\epsilon^2 = \big(\sqrt{(c_x - k_x)^2 + (c_y - k_y)^2} \pm d\big)^2.$$

The smallest value of the error is always given by the negative sign, This can be seen by contemplating the two possible cases shown in figure 3. Finally here, the square of the error can be written using the tetracyclic coordinates as follows. Let $\mathring{p}$ be the zero-radius circle centred at the solution. As we have seen above, $\mathring{p}^T J \mathring{c} = d^2 - (c_x - p_x)^2 + (c_y - p_y)^2$ and $\mathring{c}^T J \mathring{c} = 2d^2$. So,

$$\epsilon^2 = \left(\sqrt{\frac{\mathring{c}^T J \mathring{c}}{2} - \mathring{p}^T J \mathring{c}} - \sqrt{\frac{\mathring{c}^T J \mathring{c}}{2}}\right)^2.$$

This is the error for a single beacon. Adding the errors for all the beacons it is possible to perform a numerical minimisation to determine the "least squares solution".

3 Time Difference of Arrivals (TDOA)

In this section we look at a subtly different problem. We are still trying to find the location of a mobile robot but now the sensors are positioned in the

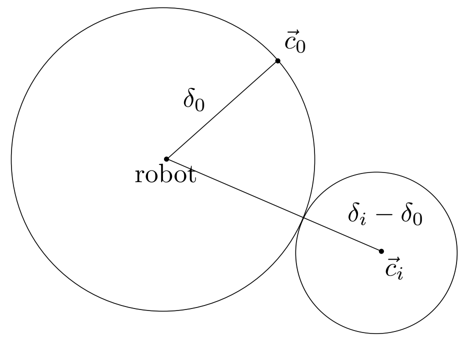


Figure 4: Time Difference of Arrival Interpreted as Touching Circles.

environment. These sensors will be referred to as "stations". Now the robot will emit a series of signals and these will be detected by the stations. Without a complicated synchronisation system there is no way to determine the time-of-flight of these signals. But we may assume that the stations are synchronised so that difference in arrival times of the signals at different stations can be found. Suppose there is a single "base station" located at $\vec{c}_0$ and the other stations are located at the points $\vec{c}_i$, where $i = 1, 2, \ldots, n$. The arrival time at station i will be denoted τ_i, if the speed of the signal is v then the distance from the robot to the station will be $\delta_i = v\tau_i$. Note that the signal could be a sound wave or an electro-magnetic wave and so v is the speed of sound in the environment or respectively the speed of light. Moreover, the environment might be air or could be underwater and so forth, so v would need to take account of that too.

3.1 Geometry of Time Differences

To understand the geometry of the situation, let's concentrate on the base station and one other station. Although we cannot know the distances from the robot to the stations we can know the differences $d_i = |\delta_i - \delta_0| = v|\tau_i - \tau_0|$. This means that a circle of radius δ_0, centred on the robot's location will meet the point $\vec{c}_0$ where the base station is located. A line drawn from the robot to $\vec{c}_i$, the position of station i, will have length $\delta_i = d_i + \delta_0$. Hence a circle of radius d_i centred on $\vec{c}_i$ will kiss the first circle centred on the robot's position, see figure 4.

If the measurements are exact we can set-up this problem using tetracyclic coordinates as follows. Suppose we have just two stations and the base station, so we have two circles one for each station $\mathring{c}_1$ and $\mathring{c}_2$ with respective radii d_1 and d_2. This gives two quadratic equations for the circle centred on the robot $\mathring{k}$,

$$(\mathring{c}_i^T J \mathring{c}_i)(\mathring{k}^T J \mathring{k}) - (\mathring{k}^T J \mathring{c}_i)^2 = \mathring{k}^T W_i \mathring{k} = 0, \qquad i = 1, 2,$$

where

$$W_i = \Big(J\mathring{c}_i \mathring{c}_i^T J - (\mathring{c}_i^T J \mathring{c}_i) J \Big),$$

as in section 2.5. Lastly, the circle centred on the robot must pass through the base station, this can be expresses using a zero-radius circle $\mathring{c}_0$ for the point $\vec{c}_0$ and then requiring the linear equation,

$$\mathring{c}_0^T J\mathring{k} = 0,$$

in $\mathring{k}$ to be satisfied. This gives us three homogeneous equations in the coordinates of $\mathring{k}$. That is, we have two quadratic and one linear equation in a $\mathbb{RP}^3$, hence, by Bézout's theorem we expect up to $2 \times 2 \times 1 = 4$ solutions.

Some of these solutions will be spurious since we are only using the squares of the time differences, not the signs of the time differences themselves. This ambiguity could be resolved in a number of ways. We could retain the signs of the time differences, this amounts to recording the order in which the signals arrive at the stations. Using this information we can dismiss the spurious solutions by examining which circles are inside and which outside the solutions for $\mathring{k}$. An alternate approach might be to add another station to the set-up. With a base station and three other stations we are looking for a circle that is tangent to three other circles. This is a classical problem in geometry know as "Apollonius's problem", see [6]. It is well know and can also be seen from the arguments above, that there are up to 8 possible solutions. However, if the measurements are exact then only one of these solutions will pass through the base station, $\vec{c}_0$.

No measurements are ever exact, so we need to look at methods that can cope with errors. A possibility that suggests itself might be to minimise the residuals. That is with a base station and n other stations we minimise the quantity,

$$\Psi = \mathring{k}^T \mathcal{W} \mathring{k} = \mathring{k}^T \left(\frac{1}{n} \sum_{i=1}^{n} W_i \right) \mathring{k},$$

subject to the constraint $\mathring{c}_0^T J\mathring{k} = 0$. Lack of space precludes further consideration of this approach here.

4 Conclusion

There is a huge literature on these location problems due to their importance in many fields. The present work is only intended to highlight the geometry of these problems with the hope that simple solutions using these geometric methods can be found and used.

The geometrical ideas presented here extend simply to higher dimensions. For example, in three dimensions spheres can be represented by pentaspherical coordinates, that is by points in an $\mathbb{RP}^4$. Further, circles in 3-D correspond to lines in $\mathbb{RP}^4$ in a natural way, see [1]. Tetracyclic coordinates also provide an ideal formalism to study Möbius transformation and more generally conformal transformations in higher dimensions. Many practical algorithms in this area use inversion in a circle to convert circles to lines (circles with infinite radius) and hence simplify problems. The group of rigid body displacements is a subgroup of the conformal group, but not a normal subgroup. However, this does mean that there is a 4-dimensional representation of $SE(2)$ which acts on these tetracyclic coordinates.

It might be useful to look at some of these problems using Lie circle (or Lie sphere) geometry, see [3]. In this approach the spheres in 3-dimensions are represented by points in a 5-dimensional projective space. The advantage of this formalism is that the relation that determines if two spheres are touching is bilinear in the Lie sphere coordinates. The disadvantage is that the points corresponding to spheres form an open set of a projective quadric variety. Moreover, the points in this geometry represent oriented spheres and for these applications that might add complications.

Other location problems are also important in robotics. In many of these it is important to find the orientation of the robot as well as its position. Perhaps the best known of these is the so called $AX = XB$ problem, originally the "hand-eye calibration problem". The variables here are elements of the rigid-body displacement group $SE(3)$. The task is to find the displacement X given measured values of A and B. However, for a solution to exist A and B must be conjugate, that is they must have the same pitch. But, as observed above, this consistency condition will usually not be satisfied by measured values. Also, since there is no bi-invariant metric on the group $SE(3)$, it is difficult to see what errors mean when comparing displacements. Yet another such problem is to find the rigid-body displacement of an object from measurements of the displacement of points on the body, see [4].

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