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On the Line Geometry of Rigid-Body Inertia

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Abstract In this work several classical ideas concerning the geometry of the inertia of a rigid body are revisited. This is done using a modern approach to screw theory. A screw, or more precisely a twist, is viewed as an element of the Lie algebra to the group of proper rigid-body displacements.

Various moments of inertia, about lines, planes and points are considered as geometrical objects resulting from least-squares problems. This allows relations between the various inertias to be found quite simply.

A brief review of classical line geometry is given, this includes an outline of the theory of the linear line complex and a brief introduction to quadratic line complexes. These are related to the geometry of the inertia of an arbitrary rigid body.

Several classical problems concerning the mechanics of rigid bodies subject to impulsive wrenches are reviewed. We are able to correct a small error in Ball's seminal treatise. The notion of spatial percussion axes is introduced and these are used to solve a problem concerning the diagonalisation of the mass matrix of a two joint robot.

Keywords Inertia · Line Geometry · Screw Theory

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1 Introduction

In this work, we revisit some rather old problems in the dynamics of rigid bodies. Some of these problems are familiar but most of them have been neglected in recent times. Using modern methods, it is possible to cast these problems into a common notation and solve them quite simply. This also allows us to find connections between the different results. We are also able to propose and solve some new problems.

Several of the problems addressed here appear in Ball's treatise, [2]. This work is well known in the mechanisms and robotics community but seems to have fallen out of favour with mathematicians and physicists. Part of the aim of this work is to reintroduce Ball's ideas to a wider audience. Along the way we are able to correct a small error in Ball's treatise.

The present work began as an attempt to understand the connection between the Painvin complex and the inertia tensor of a rigid body. This connection seem to have first appeared in [5], but see also [21]. The connection was more recently noticed by the robotics and mechanisms community in [20]. As the work progressed however, it became clear that other quadratic line complexes, notably the tetrahedral complex, play a fundamental role in the subject.

Ball, does not seem to have used much line geometry in his work even though it is clear that he was aware of the contemporary work in this area. Again the subject of line geometry is one that seems to have fallen from favour amongst mathematicians. In this case, however, this is probably because most of the problems in the subject were considered solved and the subject has been incorporated into the larger area of Algebraic Geometry.

This "falling from favour" has also occurred in Dynamics where analytical tools have superseded synthetic approaches in academic preference.

1.1 Analytical versus Synthetic Approaches in Dynamics

Analytical mechanics began with Lagrange's great treatise *Mechanique Analytique* published in 1788. Over the years, and in particular in the 20th century, Lagrange's view of mechanics has become the foremost, and almost the only way to view the subject.

However there is another way. Synthetic mechanics is older, tracing its roots back to Archimedes with its more geometrical flavour. Contrast this with the famous quotation from the preface of Lagrange's treatise: "The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure."

The analytical approach begins with a defined rigid body and aims at determining the nine inertial parameters. The nine inertial parameters are: three parameters to locate the centre of mass, while the remaining six parameters specify the inertia about the centre of mass. This approach has a unique solution and therefore is simpler than the alternative synthetic approach.

The synthetic approach normally starts with a set or subset of inertial parameters and aims at synthesizing a rigid body satisfying these parameters. Unlike the analytic problem, the synthetic problem has, in general, multiple solutions.

The synthetic problem also has close connections with mechanical design, and normally other conditions or constraints are added in order to find a suitable rigid body. Such constraints can be, for example, maximum allowable stresses, geometric alignments or workspace limitations.

As pointed out in [20], the relative ease in solving the analytic problem has unbalance the relationship between the different approaches and leads to few techniques for tackling the synthetic problem. To some extent, this unbalance has distorted the view of Dynamics in the literature.

It is perhaps not surprising then that Dynamics is nowadays a highly analytic discipline without a strong synthetic branch in the same way that say Statics or Kinematics have. It is a much easier task in Statics, than it is in Dynamics, to start with a specification and calculate the required dimensions. Perhaps this is why devices such as dextrous robot hands and multi-legged running machines, which seem inherently dynamic in their nature, are still designed with a heavy reliance on the assumption of quasi-static behaviour.

This paper intends to shed some light on the lesser discussed topic of synthesis based on inertia.

2 Lines and Screws

In this section, we discuss the concept of lines and screws and their role in the treatment of inertia.

2.1 Line Geometry

Consider a pair of points in the projective space $\mathbb{P}^3$, $\tilde{r} = (r_1 : r_2 : r_3 : r_4)$ and $\tilde{q} = (q_1 : q_2 : q_3 : q_4)$. The Plücker coordinates of the line ℓ joining these two points are given by,

$$P_{ij} = \det \begin{pmatrix} r_i & q_i \\ r_j & q_j \end{pmatrix} = r_i q_j - r_j q_i,$$

where we assume $i < j$. Notice that if a different pair of points on the line are taken then the Plücker coordinates are unchanged except that they may be all multiplied by a non-zero constant. Hence, the six coordinates can be taken as homogeneous coordinates in five dimensional projective space, $\mathbb{P}^5$. The Plücker coordinates can be collected into a vector in the following order,

$$\ell = \begin{pmatrix} P_{14} \\ P_{24} \\ P_{34} \\ P_{23} \\ P_{31} \\ P_{12} \end{pmatrix}.$$

Notice that $P_{31} = -P_{13}$ is used here, this is to avoid minus signs in some of the following equations.

However, not all points in $\mathbb{P}^5$ correspond to lines in $\mathbb{P}^3$ since their coordinates have to satisfy a condition expressed in eq. (1) below.

To obtain this condition, a simple way is to consider the following 4×4 determinant equation

$$\det\begin{pmatrix} r_1 & q_1 & r_1 & q_1 \\ r_2 & q_2 & r_2 & q_2 \\ r_3 & q_3 & r_3 & q_3 \\ r_4 & q_4 & r_4 & q_4 \end{pmatrix} = 0,$$

The determinant above is clearly zero because of the repeated columns. However, if the determinant is expanded in terms of its 2×2 sub-determinants we get,

$$P_{12}P_{34} + P_{31}P_{24} + P_{23}P_{14} = 0. \tag{1}$$

This quadratic equation defines a 4-dimensional quadric variety in $\mathbb{P}^5$ that is usually known as the Klein quadric or sometimes Plücker's quadric. The lines of $\mathbb{P}^3$ are in 1-to-1 correspondence with the points of this quadric.

In kinematics, we are usually more interested in the lines in $\mathbb{R}^3$. The points in $\mathbb{P}^3$ with homogeneous coordinates of the form $(x:y:z:1)$ can be used to represent points (x,y,z) in $\mathbb{R}^3$. Now, given a pair of points $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ in $\mathbb{R}^3$ the first three Plücker coordinates become,

$$\omega = \begin{pmatrix} P_{14} \\ P_{24} \\ P_{34} \end{pmatrix} = \begin{pmatrix} r_1 - q_1 \\ r_2 - q_2 \\ r_3 - q_3 \end{pmatrix},$$

this is a vector in the direction of the line defined by $\mathbf{r}$ and $\mathbf{q}$. It will be written as,

$$\omega = \mathbf{r} - \mathbf{q}.$$

It is convenient to write the second triple of Plücker coordinates as

$$\mathbf{v} = \begin{pmatrix} P_{23} \\ P_{31} \\ P_{12} \end{pmatrix} = \begin{pmatrix} r_2q_3 - r_3q_2 \\ r_3q_1 - r_1q_3 \\ r_1q_2 - r_2q_1 \end{pmatrix},$$

this vector is the moment of the line. This can be written,

$$\mathbf{v} = \mathbf{q} \times \mathbf{r} = \mathbf{q} \times \omega.$$

With these coordinates a line can be written as a 6-vector, partitioned as two 3-vectors,

$$\ell = \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}.$$

The equation for the Klein quadric (1), with these coordinate becomes,

$$\omega \cdot \mathbf{v} = 0. \tag{2}$$

Now the correspondence between lines in $\mathbb{R}^3$ and points in the Klein quadric is still 1-to-1 but there is an exceptional set of points in the quadric not corresponding to any physical lines. This exceptional set forms a 2-plane given by $\omega = \mathbf{0}$. Points in this 2-plane are sometimes referred to as 'lines at infinity' or 'screws of infinite pitch' as described in section 2.2 below.

2.2 Twists and Wrenches

In Ball's classic work [2] screws are defined as geometric objects similar to lines. Twists are then defined as screws with an amplitude, while wrenches are screws with an intensity. Here we take a different view more in line with current mathematical standards. A twist is a 6-dimensional vector which transforms according to the adjoint representation of the group of rigid-body displacements. That is, a twist is an element of the Lie algebra to the group of rigid displacements. Twists can be used to describe generalised velocities of rigid bodies. These 6-dimensional vectors can be partitioned into a pair of 3-vectors,

$$\mathbf{s} = \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix},$$

where ω is the angular velocity of the body and $\mathbf{v}$ is the velocity of a point in the body instantaneously located at the origin. In this view, a screw is an element of the projective space formed from the Lie algebra. It is also possible to represent lines and twist by dual vectors of the form,

$$\check{\mathbf{s}} = \omega + \varepsilon \mathbf{v},$$

there is a long tradition, beginning with Clifford, using dual numbers and dual quaternions to express problems in kinematics. Unfortunately the representation of inertias in this formalism is rather awkward and hence these ideas will not be pursued further here.

There are two natural quadratic forms or pseudo-metrics, defined on twists. The first, is the reciprocal product given by the Klein form,

$$\mathbf{s}^T Q_0 \mathbf{s} = 2\omega \cdot \mathbf{v},$$

where

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \tag{3}$$

with I_3 the 3×3 identity matrix.

The second is called the Killing form and can be defined for any Lie group, here it has the form,

$$\mathbf{s}^T Q_\infty \mathbf{s} = \omega \cdot \omega,$$

where

$$Q_\infty = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

The pitch of a twist or screw is given by,

$$\text{pitch} = \frac{\mathbf{s}^T Q_0 \mathbf{s}}{2\mathbf{s}^T Q_\infty \mathbf{s}}.$$

Notice that a line can be defined as a twist with zero pitch. The lines at infinity will have vanishing Killing and Klein forms.

As mentioned above, the action of the group of rigid displacements on twists is given by the adjoint representation of the group. In partitioned matrix form, this group action can be written as

$$\begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} \mapsto \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix},$$

where R is the usual 3×3 rotation matrix and T is the 3×3 anti-symmetric matrix corresponding to a translation vector $\mathbf{t}$. That is, if $\mathbf{t} = (t_x, t_y, t_z)^T$ then,

$$T = \begin{pmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{pmatrix}, \tag{5}$$

so that for an arbitrary vector $\mathbf{p}$ we have that $T\mathbf{p} = \mathbf{t} \times \mathbf{p}$.

Now in order that the results of the quadratic forms are scalars, that is independent of the coordinate system in which they are computed, the matrices Q_0, Q_∞, must transform under the following representation of the group,

$$\begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \mapsto \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}^{-T} \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}^{-1}$$

Computing the inverse and transpose of these matrices is quite easy, then it is simple to see that,

$$\begin{pmatrix} R & TR \\ 0 & R \end{pmatrix} \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \begin{pmatrix} R^T & 0 \\ -R^T T & R^T \end{pmatrix} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$$

and similarly,

$$\begin{pmatrix} R & TR \\ 0 & R \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R^T & 0 \\ -R^T T & R^T \end{pmatrix} = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, it is not too difficult to prove that these two forms and their linear combinations are the only forms with this invariance property, see for example [16, §6.4]. Gibson and Hunt [7], denote the linear combination of these symmetric matrices,

$$Q_h = \alpha Q_0 + \beta Q_\infty,$$

where $h = \alpha/\beta$. The matrices Q_h then define the pitch quadrics discussed in [7] .

In [2] Ball defined wrenches, the combination of force and torque vectors as screws with an intensity and then used the reciprocal product of a wrench with a twist to give the power or work rate of a wrench acting on a body moving with a twist. Here, however, wrenches are defined as 6-vectors which transform according to the co-adjoint representation of the group of rigid displacements. That is, wrenches are elements of the dual to the Lie algebra. This means that the shape of a wrench partitioned into two 3-vectors is,

$$\mathscr{W} = \begin{pmatrix} \mathbf{m} \\ \mathbf{F} \end{pmatrix}$$

where **F** is the vector in the direction of a force and **m** is a torque or moment vector. In older terminology, the twists would be called *covariant vectors* while the wrenches would be called *contravariant vectors*. Going back even further to the terminology of Plücker, we could think of the difference between twists and wrenches as being that one is written in *ray order* while the other is written in *axis order*.

The wrench associated to a pure force acting on a line through a point $\mathbf{r}$ is given by,

$$\mathscr{W} = \begin{pmatrix} \mathbf{r} \times \mathbf{F} \\ \mathbf{F} \end{pmatrix}.$$

This means that the power of a wrench $\mathscr{W}$ acting on a body moving with twist $\mathbf{s}$ is given by,

$$\text{power} = \mathscr{W}^T \mathbf{s}.$$

The co-adjoint representation of the group is given by,

$$\begin{pmatrix} \mathbf{m} \\ \mathbf{F} \end{pmatrix} \mapsto \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}^{-T} \begin{pmatrix} \mathbf{m} \\ \mathbf{F} \end{pmatrix} = \begin{pmatrix} R & TR \\ 0 & R \end{pmatrix} \begin{pmatrix} \mathbf{m} \\ \mathbf{F} \end{pmatrix}.$$

This ensures that if we compute the power in a different coordinate system the result will be the same. Suppose we subject the wrench and twist to an arbitrary rotation and translation, the power will be given by,

$$\left(\begin{pmatrix} R & TR \\ 0 & R \end{pmatrix} \begin{pmatrix} \mathbf{m} \\ \mathbf{F} \end{pmatrix} \right)^T \left(\begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} \right) =$$
$$(\mathbf{m}^T, \mathbf{F}^T) \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}^{-1} \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix} \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} = (\mathbf{m}^T, \mathbf{F}^T) \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}.$$

Notice also that the product $Q_0\mathbf{s}$ transforms according to the co-adjoint representation and is thus a wrench.

This approach to Screw Theory is explained in more detail in [16]. We now go a little deeper into the application of screw theory to inertia analysis. The first step is the computation of distances from points to geometrical entities.

3 Mean Squared Distances

In this work the geometry of the situation is the main theme; hence, we will assume that the density of the rigid-body under consideration is uniform. In this way, the position of the points comprising the body are considered but their masses are essentially irrelevant to our purposes here.

3.1 Distance to a Plane

Suppose that $\tilde{\mathbf{p}} = (p_x, p_y, p_z, 1)^T$ is a point in the rigid-body, where the coordinates p_i are measured with respect to some fixed coordinate frame. The coordinates of the point have been written as homogeneous 4-dimensional vectors so that the distance to a plane $\tilde{\pi}$ can be written as,

$$h_\pi = \tilde{\pi}^T \tilde{\mathbf{p}} = \tilde{\mathbf{p}}^T \tilde{\pi},$$

where the plane has coordinates $\tilde{\pi} = (n_x, n_y, n_z, -d)^T$, with n_i the coordinates of the unit normal vector to the plane and d is the perpendicular distance of the plane to origin of our fixed coordinate frame.

The square of the distance from the point to the plane can thus be written as,

$$h_\pi^2 = \tilde{\pi}^T \tilde{\mathbf{p}} \tilde{\mathbf{p}}^T \tilde{\pi}.$$

To find the mean-squared distance of a rigid-body to a plane $\tilde{\pi}$ we can integrate the above equation over all points in the body and divide by the volume $V_B = \int_B dV$ of the body, here $\int_B$ denotes integration over the body B and dV is the standard volume measure.

$$k_\pi^2 = \frac{\int_B h_\pi^2 dV}{\int_B dV} = \frac{1}{V_B} \tilde{\pi}^T \left(\int_B \tilde{\mathbf{p}} \tilde{\mathbf{p}}^T \, dV \right) \tilde{\pi} = \tilde{\pi}^T \tilde{\Xi} \tilde{\pi}. \tag{6}$$

Here the components of the symmetric 4×4 matrix $\tilde{\Xi}$ are

$$\tilde{\Xi} = \frac{1}{V_B} \begin{pmatrix} \int_B p_x^2 \, dV & \int_B p_x p_y \, dV & \int_B p_x p_z \, dV & \int_B p_x \, dV \\ \int_B p_x p_y \, dV & \int_B p_y^2 \, dV & \int_B p_y p_z \, dV & \int_B p_y \, dV \\ \int_B p_x p_z \, dV & \int_B p_y p_z \, dV & \int_B p_z^2 \, dV & \int_B p_z \, dV \\ \int_B p_x \, dV & \int_B p_y \, dV & \int_B p_z \, dV & \int_B dV \end{pmatrix}. \tag{7}$$

In the following, the matrix $\tilde{\Xi}$ will be referred to as the *homogeneous plane-distance inertia matrix*.

Notice that the last column of eq. (7) and, by symmetry, its final row is the homogeneous 4-vector giving the position $\mathbf{c}$ of the centroid of the body. The matrix can be partitioned as,

$$\tilde{\Xi} = \begin{pmatrix} \Xi & \mathbf{c} \\ \mathbf{c}^T & 1 \end{pmatrix}. \tag{8}$$

where the top left-hand 3×3 block Ξ, will be called the *inhomogeneous plane-distance inertia matrix*, that is,

$$\Xi = \frac{1}{V_B} \int_B \begin{pmatrix} p_x^2 & p_x p_y & p_x p_z \\ p_x p_y & p_y^2 & p_y p_z \\ p_x p_z & p_y p_z & p_z^2 \end{pmatrix} dV. \tag{9}$$

The matrix $\mathbf{p}\mathbf{p}^T$ in the above equation is sometimes called 'dyadic square' and if $\mathbf{p}$ is a unit vector the matrix represents the orthogonal projection onto the line with direction vector $\mathbf{p}$.

Consider the set of all planes in space that have the same mean-squared distance k^2 to a body. These planes, from eq. (6), clearly satisfy the quadratic equation,

$$\tilde{\pi}^T \left(\widetilde{\Xi} - k^2 \widetilde{Q}_\infty\right) \tilde{\pi} = 0, \tag{10}$$

Here the 4×4 matrix $\widetilde{Q}_\infty$ has the partitioned form,

$$\widetilde{Q}_\infty = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

For a fixed k, these planes are all tangent to a quadric in $\mathbb{P}^3$. This quadric has the equation,

$$\tilde{\mathbf{p}}^T \widetilde{\Upsilon}_k \tilde{\mathbf{p}} = 0, \tag{11}$$

where,

$$\widetilde{\Upsilon}_k = \operatorname{Adj}\left(\widetilde{\Xi} - k^2 \widetilde{Q}_\infty\right). \tag{12}$$

Here the adjugate, or adjoint, of a matrix, denoted Adj, is the transposed matrix of cofactors. That is the adjugate of a matrix M satisfies $M \operatorname{Adj}(M) = \det(M) I$, where I is the identity matrix of the appropriate dimension. In the case above I has dimension 4. Note that the tangent plane to the quadric at a point $\tilde{\mathbf{p}}$ is given by,

$$\tilde{\pi} = \widetilde{\Upsilon}_k \tilde{\mathbf{p}}. \tag{13}$$

Substituting eq. (13) into eq. (10) above yields eq. (11).

The quadrics defined by eqs. (11) are known as *isogyre quadrics*, see [21]. To say a little more about the isogyre quadrics let us choose to work in coordinates with origin located at the body's centroid and axes along the principal axes of inertia, so that the inhomogeneous plane-distance inertia matrix Ξ, eq. (8), is diagonal. In this coordinate system, the homogeneous plane-distance inertia matrix $\widetilde{\Xi}$, takes the form,

$$\widetilde{\Xi} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

where we will assume $a \geq b \geq c$. Now the matrix $\widetilde{\Upsilon}_k$ can be written,

$$\widetilde{\Upsilon}_k = \begin{pmatrix} \frac{1}{a^2-k^2} & 0 & 0 & 0 \\ 0 & \frac{1}{b^2-k^2} & 0 & 0 \\ 0 & 0 & \frac{1}{c^2-k^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that we have used the inverse here rather than the adjugate, so long as the determinant is non-zero dividing by a constant factor does not affect the quadric. We see that the family of quadrics, parametrised by k are confocal and their real type can be found from the signature of the matrix $\widetilde{\Upsilon}_k$, see Table 1. See also [21] and [15, Art.158] or [19, Chap. VIII] or [8]. It is also interesting to note that the idea of confocal quadrics is central to the work of Ivory on the gravitational attraction of ellipsoids [10], this led to Ivory's theorem often used in kinematics.

Table 1 Real Type of the Isogyre Quadrics.

condition	type	signature of $\widetilde{\Upsilon}_k$
$k < c$	imaginary	$++++$
$c < k < b$	hyperboloids of two sheets	$+++-$
$b < k < a$	hyperboloids of one sheet	$++--$
$a < k$	ellipsoids	$+---$

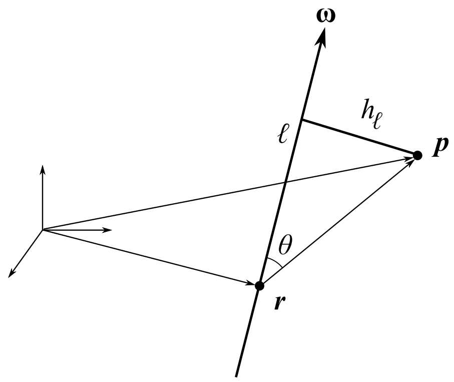

Fig. 1 Distance from a point to line

3.2 Distance to a Line

Next we look at the mean-squared distance of points in a body to a line. Plücker coordinates will be used to refer to lines, these six-dimensional vectors can be partitioned as a pair of three vectors,

$$\ell = \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}, \tag{15}$$

where ω is a unit vector in the direction of the line and $\mathbf{v}$ is the moment of the line see section 2.1 above. Also points here will be specified by 3-vectors $\mathbf{p}^T = (p_x, p_y, p_z)$.

To begin with, let $\mathbf{r}$ be any point on the line ℓ. The perpendicular distance h_ℓ from an arbitrary point $\mathbf{p}$ to the line is given by,

$$\begin{aligned} h_\ell &= |\mathbf{p}-\mathbf{r}|\sin\theta \\ &= |(\mathbf{p}-\mathbf{r})\times\omega| = |(\mathbf{p}\times\omega)-\mathbf{v}|, \end{aligned} \tag{16}$$

where θ is the angle between the line and the vector $\mathbf{p}-\mathbf{r}$ and $\mathbf{v} = \mathbf{r}\times\omega$ is the moment of the line, see Fig. 1. Now let P be the 3×3 antisymmetric matrix corresponding to the vector product with the vector $\mathbf{p}$, as in equation (5) above

$$P = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix}.$$

The square of the distance h_ℓ between the point and the line, eq. (16), can then be written as

$$\begin{aligned} h_\ell^2 &= (P\omega - I_3\mathbf{v})^T(P\omega - I_3\mathbf{v}) \\ &= \omega^T P^T P\omega - \omega^T P^T\mathbf{v} - \mathbf{v}^T P\omega + \mathbf{v}^T I_3\mathbf{v}. \end{aligned} \tag{17}$$

In a partitioned matrix form, eq. (17) can be written as

$$h_\ell^2 = (\omega^T, \mathbf{v}^T)\begin{pmatrix} P^T P & P \\ P^T & I_3 \end{pmatrix}\begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}.$$

To find the mean-square distance from the line to the point in a rigid-body we can integrate over the body and divide by the total volume as usual, to get,

$$k_\ell^2 = \frac{1}{V_B}\int_B h_\ell^2\, dV = \ell^T N\ell, \tag{18}$$

where N is the usual 6×6 rigid-body inertia matrix — assuming the body has uniform density and unit mass. Slightly more generally, if the rigid body has uniform density but total mass m then mN would be the inertia matrix for the body.

The matrix N has the partitioned form,

$$N = \begin{pmatrix} \mathbf{I} & C \\ C^T & I_3 \end{pmatrix} = \frac{1}{V_B}\begin{pmatrix} \int_B P^T P\, dV & \int_B P\, dV \\ \int_B P^T\, dV & \int_B I_3\, dV \end{pmatrix}. \tag{19}$$

Here $\mathbf{I}$ is the standard 3×3 inertia matrix

$$\mathbf{I} = \frac{\int_B P^T P\, dV}{V_B} \tag{20}$$

which has components $\mathbf{I}_{xx} = \int_B (p_y^2 + p_z^2)\, dV/V_B$, $\mathbf{I}_{xy} = -\int_B p_x p_y\, dV/V_B$ and so forth. The matrix C in eq. (19) is the anti-symmetric matrix corresponding to the centroid $\mathbf{c}$, of the body. The quantity k_ℓ in eq. (18) is the radius of gyration of the body about the line ℓ.

3.3 Distance to a Point

Consider the problem of finding the mean squared distance from the points of the body $\mathbf{p}$ to a point $\tilde{\mathbf{r}}^T = (\mathbf{r}^T, 1)$ where $\mathbf{r} = (r_x, r_y, r_z)^T$. The squared distance h_r^2 from $\mathbf{r}$ to a general point $\mathbf{p}$ in the body can be written,

$$h_r^2 = |\mathbf{r} - \mathbf{p}|^2 = (\mathbf{r} - \mathbf{p})^T(\mathbf{r} - \mathbf{p}) = \mathbf{r}^T\mathbf{r} - \mathbf{r}^T\mathbf{p} - \mathbf{p}^T\mathbf{r} + \mathbf{p}^T\mathbf{p}.$$

Which can be written in the following matrix form,

$$h_r^2 = (\mathbf{r}^T, 1)\begin{pmatrix} I_3 & -\mathbf{p} \\ -\mathbf{p}^T & \mathbf{p}^T\mathbf{p} \end{pmatrix}\begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix}.$$

Integrating h_r^2 and dividing by the volume of the body as in the previous two sections gives,

$$k_r^2 = \frac{\int_B h_r^2 dV}{\int_B dV} = (\mathbf{r}^T, 1) \begin{pmatrix} I_3 & -\mathbf{c} \\ -\mathbf{c}^T & \sigma^2 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix}, \tag{21}$$

where $\sigma^2 = \int_B \mathbf{p} \cdot \mathbf{p}\, dV/V_B$ and $\mathbf{c}$ is the position of the centroid, as before.

The matrix,

$$\widetilde{\Sigma} = \begin{pmatrix} I_3 & -\mathbf{c} \\ -\mathbf{c}^T & \sigma^2 \end{pmatrix},$$

could be called the polar inertia matrix and we could refer to σ^2 as the moment of inertia about the centroid.

Finally here, it is worth observing that it is clear from the definitions of the matrices, $\widetilde{\Sigma}$, $\widetilde{\Xi}$ and N, that they are positive definite matrices. They are sums of squares of various distances. The fact that the 3×3 inertia matrix $\mathbf{I}$ is positive definite is often inferred from physical considerations—the kinetic energy of the body must be positive. However, here we see that these facts can be shown using the geometry of the situation only.

3.4 Relations between Distances

There are several relations between the matrices and quantities derived above. These will be studied in this section.

Suppose μ and ν are two perpendicular planes with intersection line ℓ, then by Pythagoras' theorem it is clear that,

$$h_\ell^2 = h_\mu^2 + h_\nu^2,$$

see Fig. 2. Here h_ℓ is the perpendicular distance from a point in the body to the line ℓ; h_μ is the perpendicular distance to the plane μ and similar for h_ν. Integrating this equation over the body and dividing by the volume of the body gives,

$$k_\ell^2 = k_\mu^2 + k_\nu^2. \tag{22}$$

A similar result can be produced using the 3-dimensional version of Pythagoras' theorem. Suppose we have 3 mutually orthogonal planes $\tilde{\pi}_1$, $\tilde{\pi}_2$ and $\tilde{\pi}_3$ and suppose these planes meet at a point $\mathbf{r}$. Then the perpendicular distances from an arbitrary point $\mathbf{p}$ to the three planes are given by the length of the vector from $\mathbf{r}$ to $\mathbf{p}$ multiplied by a direction cosine. Hence, after integrating over the body we have the relation,

$$k_{\pi_1}^2 + k_{\pi_2}^2 + k_{\pi_3}^2 = k_r^2. \tag{23}$$

Multiply the above equation by two, now for each pair of planes apply equation (22) above. Writing $k_{\ell_{12}}$ for the radius of gyration about the intersection of the planes π_1 and π_2 and so forth, we obtain the relation,

$$k_{\ell_{12}}^2 + k_{\ell_{23}}^2 + k_{\ell_{13}}^2 = 2k_r^2. \tag{24}$$

Notice that this relation will apply to any three mutually orthogonal lines ℓ_{12}, ℓ_{23}, ℓ_{13} meeting at a point $\mathbf{r}$.

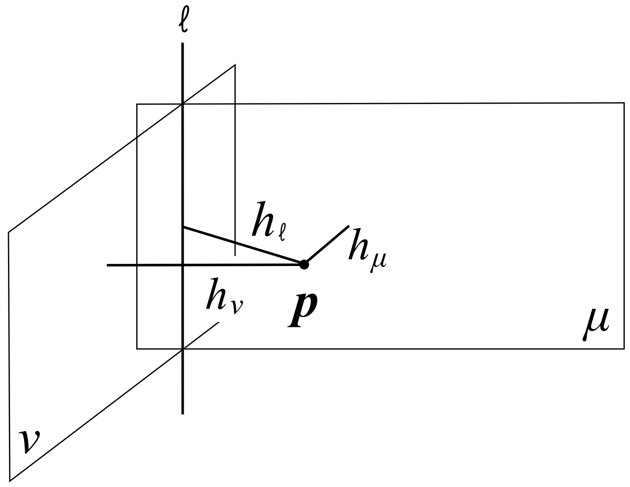

Fig. 2 Pythagoras' Theorem showing distances h_μ, h_ν and h_ℓ from a point $\mathbf{p}$ to two perpendicular planes μ and ν and their intersection line ℓ.

Next consider the following relation for a general 3-vector,

$$P^T P = (\mathbf{p}^T \mathbf{p}) I_3 - \mathbf{p}\mathbf{p}^T,$$

where, as usual, P is the 3×3 anti-symmetric matrix representing the vector $\mathbf{p}$. This equation can be easily verified by direct computation. Integrating the relation over the body and dividing by its volume yields an equation relating the various inertia matrices we have found above,

$$\mathbf{I} + \Xi = \sigma^2 I_3, \tag{25}$$

where $\mathbf{I}$ is the standard 3×3 inertia matrix defined in (20), Ξ is the inhomogeneous plane-distance inertia matrix defined in equation (9) and σ^2 is the moment of inertia about the centroid.

Consider a point $\mathbf{r}$ located on a plane $\tilde{\pi}$ together with the line normal to the plane through the point, ℓ. Now we may assume that μ and ν are two mutually orthogonal planes which meet along the line ℓ. Hence these planes contain $\mathbf{r}$ and are both orthogonal to $\tilde{\pi}$. Then by combining equations (22) and (23) to eliminate k_μ^2 and k_ν^2 we get the result,

$$k_\ell^2 + k_\pi^2 = k_r^2. \tag{26}$$

Note that this result also appears in [21, p.12].

Now again choose three mutually orthogonal planes but now suppose that these planes are normal to the three principal inertia directions meeting at the centroid of the body. Using the equation (23) above we can see that,

$$a^2 + b^2 + c^2 = \sigma^2, \tag{27}$$

where a^2, b^2, c^2 are the eigenvalues of the inhomogeneous plane-distance inertia matrix Ξ as in (14) above, and σ^2 is the moment of inertia about the centroid as in (21) above.

Notice that equation (25) above implies that when the inertia matrix $\mathbf{I}$ is diagonal then so is Ξ. So if we choose coordinates with origin at the centroid of the body and aligned with the principal axes of inertia we can assume that, $\mathbf{I} = \mathrm{diag}(k_1^2, k_2^2, k_3^2)$ and $\Xi = \mathrm{diag}(a^2, b^2, c^2)$. Now equation (22) implies that,

$$k_1^2 = b^2 + c^2 \;=\; \sigma^2 - a^2, \tag{28}$$
$$k_2^2 = a^2 + c^2 \;=\; \sigma^2 - b^2, \tag{29}$$
$$k_3^2 = a^2 + b^2 \;=\; \sigma^2 - c^2, \tag{30}$$

since, for example, k_1^2 is the moment of inertia about the x-axis while b^2 and c^2 are the moments of inertia about the xz and xy-planes respectively.

These equations are easily inverted to give,

$$a^2 \;=\; \frac{1}{2}(k_2^2 + k_3^2 - k_1^2) \;=\; \sigma^2 - k_1^2, \tag{31}$$
$$b^2 \;=\; \frac{1}{2}(k_3^2 + k_1^2 - k_2^2) \;=\; \sigma^2 - k_2^2, \tag{32}$$
$$c^2 \;=\; \frac{1}{2}(k_1^2 + k_2^2 - k_3^2) \;=\; \sigma^2 - k_3^2. \tag{33}$$

By invoking equation (24) or by adding the equations (28)–(30) above we have,

$$k_1^2 + k_2^2 + k_3^2 = 2\sigma^2.$$

Next we investigate some of the inequalities satisfied by these quantities. Consider equation (22) above, notice that since the quantities in the equation are all positive, if ℓ is any line lying in a plane μ then we have the inequality,

$$k_\ell^2 > k_\mu^2. \tag{34}$$

This can be extended by considering equation (26),

$$k_r^2 > k_\ell^2 > k_\mu^2, \tag{35}$$

where $\mathbf{r}$ is a point lying on a line ℓ which in turn lies on a plane μ.

Now suppose that we have three mutually orthogonal lines ℓ_1, ℓ_2, ℓ_3 all passing through a common point. Let μ_{12} be the plane defined by ℓ_1 and ℓ_2 and similarly let μ_{13} be defined by ℓ_1 and ℓ_3. So by equation (22) above,

$$k_{\ell_1}^2 = k_{\mu_{12}}^2 + k_{\mu_{13}}^2. \tag{36}$$

Then using equation (22) twice more we get,

$$k_{\ell_1}^2 = k_{\ell_2}^2 + k_{\ell_3}^2 - 2k_{\mu_{23}}^2. \tag{37}$$

As an inequality we can write,

$$k_{\ell_1}^2 < k_{\ell_2}^2 + k_{\ell_3}^2. \tag{38}$$

This is a familiar result, usually applied to the principal inertias of a rigid body. Notice however, that the result does not really depend on the lines being mutually orthogonal and concurrent, all that is required ℓ_2 and ℓ_3 lie in orthogonal planes which intersect in ℓ_1.

On the other hand, if the three lines are mutually orthogonal and concurrent then the inequality applies to any cyclic permutation of the lines. The inequality is often referred to as the triangle inequality. Since if the principal inertias $k^2_{\ell_1}$, $k^2_{\ell_2}$ and $k^2_{\ell_3}$, are thought of as the lengths of line segments then the inequality means that these line segments can be arranged to form a triangle. Further, regarding equation (37) as the Cosine rule for the triangle, we can see that the triangle is acute, that is all the triangle's internal angles will be acute.

There are several other inequalities that can be derived. For example, with the same arrangement of lines and plane as above it is possible to show that,

$$k^2_{\mu_{12}} = \frac{1}{2}(k^2_{\ell_1} + k^2_{\ell_2} - k^2_{\ell_3}). \tag{39}$$

Hence we get an inequality,

$$k^2_{\ell_1} + k^2_{\ell_2} > 2k^2_{\mu_{12}}, \tag{40}$$

where ℓ_1 and ℓ_2 are a pair of perpendicular lines lying in the plane μ_{12}.

In section 3.1 it was assumed that the eigenvalues of the inhomogeneous plane-distance inertia matrix Ξ satisfy $a^2 \geq b^2 \geq c^2$. With this assumption equations (28)–(30) above imply that $k_3^2 \geq k_2^2 \geq k_1^2$. By manipulating inequalities the four possible cases for the form of the isogyre quadrics can be written in terms of the principal moments of inertia, see Table 2.

Table 2 Real Type of the Isogyre Quadrics from Principal Inertias.

condition k,a,b,c	condition k,k_1,k_3,k_3,σ	type
$k^2 < c^2$	$(\sigma^2 - k^2) > k_3^2$	imaginary
$c^2 < k^2 < b^2$	$k_3^2 > (\sigma^2 - k^2) > k_2^2$	hyperboloids of two sheets
$b^2 < k^2 < a^2$	$k_2^2 > (\sigma^2 - k^2) > k_1^2$	hyperboloids of one sheet
$a^2 < k^2$	$(\sigma^2 - k^2) > k_1^2$	ellipsoids

4 Line Complexes

A line complex is the 3-dimensional set of lines formed by the intersection of the Klein quadric with another hypersurface in $\mathbb{P}^5$. The intersection of the Klein quadric with a hyperplane gives a linear line complex, the intersection with a quadric hypersurface gives a quadratic line complex and so forth.

4.1 Linear Line Complexes

A linear line complex can be specified by a homogeneous linear equation in the Plücker coordinates of the line. In partitioned form this could be written as,

$$\mathbf{m}\cdot\omega+\mathbf{F}\cdot\mathbf{v}=0.$$

The coefficients here can be thought of a the components of a wrench,

$$\mathscr{W}^T\ell=(\mathbf{m}^T,\mathbf{F}^T)\begin{pmatrix}\omega\\ \mathbf{v}\end{pmatrix}=0.$$

That is the lines in a linear line complex can be thought of as the set of lines dual to a given wrench. On the other hand the linear equation could arise as the reciprocal product with a given twist,

$$\mathbf{s}^TQ_0\ell=0.$$

Hence, such a line complex can also be thought of as the set of lines reciprocal to a given twist. Actually the amplitude of the twist (or wrench) is immaterial since the equation is homogeneous, hence we talk about the complex of lines reciprocal to a given screw. Indeed, linear line complexes were sometimes referred to screws in older literature, see for example [4] and the postscript to [1].

This also means that many of the properties of screws carry over to linear line complexes. So a linear line complex will have an axis, usually called the central axis of the complex, and a pitch.

The lines in a linear complex have a helical distribution about the central axis of the complex. Suppose the pitch of a linear complex is p, and consider a line in the complex whose direction makes an angle ϕ with the direction of the central axis. Now the perpendicular distance between the line and the central axis d, will satisfy,

$$d=-\frac{p}{\tan\phi}.$$

Therefore, the smaller the separation between the lines the closer the angle between them gets to $\pm\pi/2$; conversely, the larger the distance between the lines the closer the angle between them gets to 0.

If the pitch of the complex is zero, the complex is called a special linear complex. Representing a linear complex as the intersection of the Klein quadric with a hyperplane in $\mathbb{P}^5$, the special linear complex is distinguished by the fact that the hyperplane is tangent to the Klein quadric. The point at which the hyperplane is tangent to the quadric corresponds to a line in $\mathbb{P}^3$, in particular this line is the central axis of the special linear complex. In fact it is not hard to see that the special linear complex consists of the set of all lines meeting a given line. This includes parallel lines which can be though of the lines meeting the given line at infinity.

In kinematics, two linear complexes arise very naturally. For each complex there are several possible geometric constructions. For the first consider a general point $\tilde{\mathbf{p}}$ in $\mathbb{P}^3$, now subject this point to a rigid displacement and suppose the image of the point is $\tilde{\mathbf{q}}$. Now the set of lines through the mid-point of the line-segment joining $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ but normal to the line form a plane-star of lines. The set of all these plane-stars for each point $\tilde{\mathbf{p}}$ form a linear line complex. In the case that the

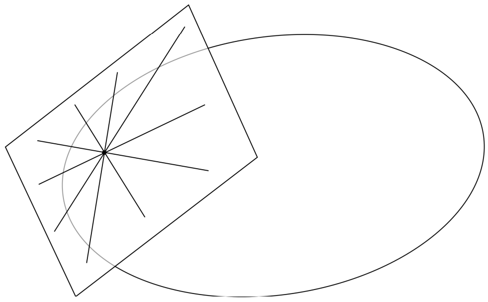

Fig. 3 The special quadratic complex: for any point on a quadric the set of tangent lines to the surface lie in the complex.

body is undergoing a rigid motion the normals to the velocities of point also form a linear complex. This result appears in Chapter V, Note B of [6], see also [14].

The second linear line complex associated with a rigid-body displacement can be described using three points. Assume that $\tilde{\mathbf{r}}$ is the image under the rigid displacement of $\tilde{\mathbf{q}}$ which in turn is the image of $\tilde{\mathbf{p}}$. Now consider the line through the point $\tilde{\mathbf{q}}$ normal to the plane determined by the three points. The set of all lines formed in this way comprise a linear line complex, this line complex is different from the one described above, given the same rigid displacements the two complexes will have the same central axes but different pitches. More details on these complexes and different ways of describing them can be found, for example, in [3].

4.2 Quadratic Line Complexes

A quadratic line complex consists of the intersection of a pair of quadrics in $\mathbb{P}^5$; the Klein quadric, given by the 6×6 matrix Q_0, and another quadric whose symmetric 6×6 matrix we will be denoted by Q. Notice that, a quadratic line complex is more precisely defined by a pencil of quadrics. The intersection, that is the set of zeros of the two quadrics Q_0 and Q, will lie in the zero set of any linear combination of the quadrics $Q' = Q+\lambda Q_0$, where λ is a constant. So the intersection could be defined by Q_0 and Q', for example.

Since points in the Klein quadric correspond to lines in $\mathbb{P}^3$, the intersection of the two quadrics will correspond to a 3-dimensional subset of lines in $\mathbb{P}^3$. Classically, different quadratic line complexes were defined by different geometric constructions in $\mathbb{P}^3$. For example, given a quadric surface in $\mathbb{P}^3$, the set of tangent lines to the surface form a quadratic line complex. This is known as the special quadratic complex[11, p. 130], see Fig. 3.

There is a classification of quadratic line complexes due to Weiler. There are 11 classes of quadratic complex determined by the projective type of the pencil of quadrics determining the complex. Each type is conveniently described by the Segre symbol of the matrix Q_0Q, (notice that $Q_0^{-1} = Q_0$). So for example most general quadratic line complex has Segre symbol, [111111] indicating that the matrix Q_0Q for this complex has distinct eigenvalues. The special quadratic complex, mentioned above, has Segre symbol $[(111)(111)]$, this means that the matrix Q_0Q for this complex has only 2 distinct eigenvalues but that the matrix is diagonalisable. Later we will meet a complex with Segre symbol $[(22)(11)]$, this means that the matrix Q_0Q for this complex has only two distinct eigenvalues, in the Jordan normal form of Q_0Q there will be two 2×2 blocks for the first eigenvalue and two 1×1 blocks for the second.

The eleven different types can sometimes be further subdivided, there may be special relations between the eigenvalues of Q_0Q. A full account is given in [11, Art. 161–214].

One of the main features of a quadratic line complex is its singular surface. This is a surface in $\mathbb{P}^3$ closely connected with the complex. It will be described in more detail in section 4.5, below. Here we simply observe that if two complexes have different singular surfaces then they cannot be isomorphic.

4.3 The Tetrahedral Line Complex

Next we consider the tetrahedral complex, a type of line complex studied by Reye and often named for him, see [11, p. 114]. This complex has Segre symbol $[(11)(11)(11)]$ and many known geometric characterisations.

One characterisation of the tetrahedral complex is as the set of lines which meet 4 given planes in 4 points which have a constant cross-ratio. Another geometric characterisation of the tetrahedral complex is as the set of normal lines to a family of confocal quadrics.

As an example, we look at the normals to the family of confocal quadrics defined by the isogyre quadrics found above. Recall that the tangent plane $\tilde{\boldsymbol{\pi}}$, to a k-isogyre at a point $\tilde{\mathbf{p}}$ is given by, equation (13),

$$\tilde{\boldsymbol{\pi}} = \widetilde{\Upsilon}_k\tilde{\mathbf{p}}.$$

with $\widetilde{\Upsilon}_k$ defined in eq. (12). This can be rearranged to give

$$\left(\widetilde{\Xi} - k^2\widetilde{Q}_\infty\right)\tilde{\boldsymbol{\pi}} = \tilde{\mathbf{p}}, \tag{41}$$

Explicitly the components of equation (41) are

$$\begin{pmatrix} a^2-k^2 & 0 & 0 & 0 \\ 0 & b^2-k^2 & 0 & 0 \\ 0 & 0 & c^2-k^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ -d \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}. \tag{42}$$

Now the normal to the point $\tilde{\mathbf{p}}$ will be directed along the normal to the tangent plane, hence the normal line will have Plücker coordinates, $(\mathbf{n}, \mathbf{p}\times\mathbf{n}) = (n_1, n_2, n_3, p_2n_3 - p_3n_2, p_3n_1 - p_1n_3, p_1n_2 - p_2n_1)$. Using the eq. (42), we can substitute for the coordinates p_1, p_2 and p_3 of the point, so that the moment of the line is given by,

$$\mathbf{p}\times\mathbf{n} = \begin{pmatrix} p_2n_3 - p_3n_2 \\ p_3n_1 - p_1n_3 \\ p_1n_2 - p_2n_1 \end{pmatrix} = \begin{pmatrix} (b^2-c^2)n_2n_3 \\ (c^2-a^2)n_1n_3 \\ (a^2-b^2)n_1n_2 \end{pmatrix}.$$

Now it is easy to see that,

$$\mathbf{n}^T \Xi(\mathbf{p}\times\mathbf{n}) = (n_1, n_2, n_3)\begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}\begin{pmatrix} (b^2-c^2)n_2n_3 \\ (c^2-a^2)n_1n_3 \\ (a^2-b^2)n_1n_2 \end{pmatrix} = 0,$$

where Ξ is the inhomogeneous plane-distance inertia matrix in the principal axes of inertia coordinate system defined in eqs. (9) and (14) above.

Hence, writing a general line as $\ell^T = (\boldsymbol{\omega}^T, \mathbf{v})^T = \left(\mathbf{n}^T, (\mathbf{p}\times\mathbf{n})^T\right)$ it is clearly a normal to one of the isogyre quadrics if it satisfies the quadratic equation,

$$(\boldsymbol{\omega}^T, \mathbf{v})^T \begin{pmatrix} 0 & \Xi \\ \Xi & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = 0 \tag{43}$$

The tetrahedral line complex, or a closely related complex, is also important in kinematics. Consider a point $\tilde{\mathbf{p}}$, in $\mathbb{P}^3$ and subject the point to a rigid-body displacement. Call the image point $\tilde{\mathbf{q}}$, now join $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ with a line. Subjecting all points in $\mathbb{P}^3$ to the same rigid-body displacement, the lines produced form a quadratic line complex, this complex has Segre symbol $[(22)(11)]$, it could be considered a special type of tetrahedral complex. In the limit that the displacement becomes infinitesimal the lines in the complex align with the velocities of the points. This complex is still of the same type and has some applications to optical flow in computer vision, see [17]. Further, if the displacement is a more general linear transformation, not just a rigid displacement, then the corresponding line complex will be a tetrahedral complex, [11, Art. 96].

Consider an arbitrary rigid-body motion at some instant. At each point in space, construct the line through the point in the direction of the point's acceleration vector. This set of lines also forms a tetrahedral complex, see [3, Chap. 6, §12].

4.4 The Painvin Complex

The Painvin complex can be defined geometrically as follows. Consider a quadric in $\mathbb{P}^3$, the Painvin complex consists of the lines formed by the intersection of pairs of orthogonal tangent planes to a quadric, see Fig. 4.

Now consider a rigid body with a 6×6 inertia matrix N, see eqs. (18) and (19). The set of lines ℓ in space with radius of gyration $k_\ell = k$ is given by the equation,

$$\ell^T N \ell = k^2.$$

This equation is true if we assume that the lines are unit lines. That is, the lines ℓ have directions $\boldsymbol{\omega}$ satisfying $\boldsymbol{\omega}^T\boldsymbol{\omega} = 1$, see equation (15). If we assume the lines are not necessarily unit lines, then the relevant equation would be,

$$\ell^T N \ell = k^2 \boldsymbol{\omega}^T \boldsymbol{\omega}$$

which could also be written,

$$\ell^T N \ell = k^2 \ell^T Q_\infty \ell,$$

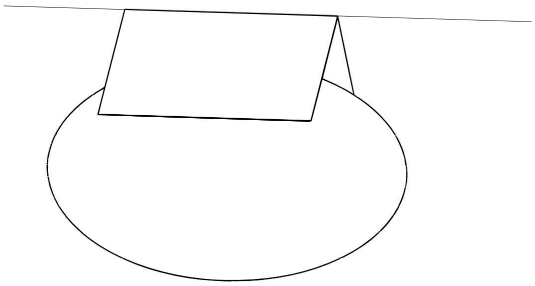

Fig. 4 The Painvin complex: the intersection of perpendicular tangent planes to a quadric gives a line of the complex.

where Q_∞ is the 6×6 invariant matrix introduced in eq. (4) of Section 2.2.

This is now a homogeneous quadratic equation in the Plücker coordinates of the lines, which could be written,

$$\ell^T(N-k^2Q_\infty)\ell = 0. \tag{44}$$

The lines satisfying the quadratic equation (44) are called *isogyre lines*. Hence the isogyre lines, for some particular value of k, form a quadratic line complex.

By considering the Plücker coordinates as homogeneous coordinates in a projective space $\mathbb{P}^5$, the solutions to this equation, taken with the equation for the Klein quadric, (2), may include some unphysical solutions for which $\omega^T\omega = 0$. The infinite lines in this quadratic complex are not real. To see this substitute $\ell^T = (\mathbf{0}^T, \mathbf{v}^T)$ into equation (44) above. For any k^2 the equation reduces to $\mathbf{v}\cdot\mathbf{v} = 0$ which clearly has no real solutions.

It is now possible to show that this complex of isogyre lines, is in fact a Painvin complex.

Theorem 1 *(Demoulin 1892) The complex of k-isogyre lines is the Painvin complex to the $(k/\sqrt{2})$-isogyre quadric.*

Recall that the k-isogyre quadric was written $\widetilde{\Upsilon}_k$ and defined by eqs. (11), (12) and (14) in section 3.1 above. Further recall that the tangent planes to this quadric all have mean-squared distance to the body $k^2/2$. Hence, by equation (22), the line of intersection between a pair of orthogonal tangent planes will have mean-squared distance to the body,

$$\frac{k^2}{2}+\frac{k^2}{2} = k^2.$$

So every line in the Painvin complex is a k-isogyre line.

It is also possible to establish the connection between the Painvin complex of an arbitrary quadric in $\mathbb{P}^3$ and the quadric satisfied by the lines of the complex.

Consider an arbitrary quadric in $\mathbb{P}^3$ given by

$$\tilde{\mathbf{p}}^T\widetilde{Q}\tilde{\mathbf{p}} = 0.$$

The polar quadric, that is, the quadric satisfied by the tangent planes to the original quadric will be given by,

$$\tilde{\pi}^T \widetilde{A} \tilde{\pi} = 0, \tag{45}$$

where the matrix $\widetilde{A} = \mathrm{Adj}(\widetilde{Q})$. It is convenient to partition this 4×4 symmetric matrix in the following pattern,

$$\mathrm{Adj}(\widetilde{Q}) = \widetilde{A} = \begin{pmatrix} A_1 & \mathbf{a}_2 \\ \mathbf{a}_2^T & a_0 \end{pmatrix}.$$

Now let,

$$\tilde{\pi}_1 = \begin{pmatrix} \mathbf{n}_1 \\ -d_1 \end{pmatrix} \quad \text{and} \quad \tilde{\pi}_2 = \begin{pmatrix} \mathbf{n}_2 \\ -d_2 \end{pmatrix},$$

be two tangent planes to the original quadric. Hence they both satisfy equation (45) above,

$$(\mathbf{n}_i^T, -d_i) \begin{pmatrix} A_1 & \mathbf{a}_2 \\ \mathbf{a}_2^T & a_0 \end{pmatrix} \begin{pmatrix} \mathbf{n}_i \\ -d_i \end{pmatrix} = \mathbf{n}_i^T A_1 \mathbf{n}_i - 2d_i(\mathbf{n}_i \cdot \mathbf{a}_2) + d_i^2 a_0 = 0, \quad i = 1, 2. \tag{46}$$

The Plücker coordinate of the line determined by the intersection of these planes is given by,

$$\ell = \begin{pmatrix} \mathbf{n}_1 \times \mathbf{n}_2 \\ d_2\mathbf{n}_1 - d_1\mathbf{n}_2 \end{pmatrix}.$$

Now consider the product,

$$\big((\mathbf{n}_1 \times \mathbf{n}_2)^T, d_2\mathbf{n}_1^T - d_1\mathbf{n}_2^T\big) \begin{pmatrix} \mathrm{Tr}(A_1)I_3 - A_1 & A_2 \\ A_2^T & a_0 I_3 \end{pmatrix} \begin{pmatrix} \mathbf{n}_1 \times \mathbf{n}_2 \\ d_2\mathbf{n}_1 - d_1\mathbf{n}_2 \end{pmatrix}$$

Assuming that the two tangent planes are perpendicular, this product simplifies to give,

$$(\mathbf{n}_1^T A_1 \mathbf{n}_1 - 2d_1(\mathbf{n}_1 \cdot \mathbf{a}_2) + d_1^2 a_0)(\mathbf{n}_2 \cdot \mathbf{n}_2) + (\mathbf{n}_2^T A_1 \mathbf{n}_2 - 2d_2(\mathbf{n}_2 \cdot \mathbf{a}_2) + d_2^2 a_0)(\mathbf{n}_1 \cdot \mathbf{n}_1),$$

this expression vanishes by virtue of equation (46).

The computation here is lengthy but straightforward, it is facilitated by the identity,

$$\begin{aligned} &(\mathbf{n}_1 \times \mathbf{n}_2)^T \Big(\mathrm{Tr}(A)I_3 - A\Big)(\mathbf{n}_1 \times \mathbf{n}_2) = \\ &\quad (\mathbf{n}_1^T A \mathbf{n}_1)(\mathbf{n}_2 \cdot \mathbf{n}_2) + (\mathbf{n}_2^T A \mathbf{n}_2)(\mathbf{n}_1 \cdot \mathbf{n}_1) - 2(\mathbf{n}_1^T A \mathbf{n}_2)(\mathbf{n}_1 \cdot \mathbf{n}_2), \end{aligned}$$

which is valid for arbitrary vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and arbitrary 3×3 symmetric matrix A.

Hence it is clear that the lines in the Painvin complex satisfy the quadratic equation, $\ell^T Q_P \ell = 0$ where the 6×6 symmetric matrix Q_P is given by,

$$Q_P = \begin{pmatrix} \mathrm{Tr}(A_1)I_3 - A_1 & A_2 \\ A_2^T & a_0 I_3 \end{pmatrix}.$$

This shows the converse to the theorem above:

Theorem 2 *(Wolkowitsch 1952) Any Painvin complex is the complex of k-isogyre lines for some rigid-body.*

Proof For this to be true the mass of the body must be positive, now the matrix for the complex can be Q_p or $-Q_P$ since the equation defining the complex is homogeneous so we can always assume that $m = a_0$ is positive and hence corresponds to the mass of the body. The centre of mass of the body will be located at $\mathbf{c} = (1/a_0)\mathbf{a}_2$. Finally the 3×3 inertia matrix of the body will be, $\mathbf{I} = \big(\mathrm{Tr}(A_1) + a_0 k^2\big) I_3 - A_1$. This must be a positive definite matrix, hence if the matrix $\mathrm{Tr}(A_1)I_3 - A_1$ is not positive definite and its smallest eigenvalue is $\lambda < 0$, we must choose k so that $a_0 k^2 > |\lambda|$. □

4.5 Singular Surfaces of Quadratic Complexes

Consider the set of lines in an arbitrary quadratic line complex which pass through a point. Suppose the matrix of the quadric defining the complex has the partitioned form,

$$Q = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

recall that Q, and hence A and C, must be symmetric. An arbitrary line through the point $\mathbf{p}$ can be written,

$$\ell = \begin{pmatrix} \omega \\ \mathbf{p} \times \omega \end{pmatrix} = \begin{pmatrix} I_3 \\ P \end{pmatrix} \omega,$$

where as usual P is the 3×3 anti-symmetric matrix corresponding to $\mathbf{p}$. The equation for the complex can now be written,

$$\omega^T \Big(I_3, P^T \Big) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} I_3 \\ P \end{pmatrix} \omega = 0,$$

or

$$\omega^T \Big(A + BP + P^T B^T + P^T CP \Big) \omega = 0.$$

With P fixed this is a quadratic equation in the components of ω. Hence the lines in the complex through point $\mathbf{p}$ form a quadratic cone with vertex at $\mathbf{p}$.

At some points in space however, the cone will degenerate to a pair of planes. This will be determined by the vanishing of the determinant,

$$\det \Big(A + BP + P^T B^T + P^T CP \Big) = 0.$$

This turns out to be a quartic polynomial in the coordinates of $\mathbf{p}$. In general, that is for a complex with Segre symbol $[111111]$, the quartic surface determined by the vanishing of this polynomial is the celebrated Kummer surface, see [9]. However, for other types of complexes different surfaces arise.

For example, for a tetrahedral complex, the surface of singular points degenerates to 4 planes. In the case of the particular tetrahedral complex discussed in section 4.3 above, these planes are the planes $x = 0$, $y = 0$, $z = 0$ and the plane at infinity $w = 0$. So in general, three of the singular planes pass through the centroid of the body and are perpendicular to one of the three principal axes of inertia. The fourth singular plane will always be the plane at infinity. In section 4.3 it was mentioned that a tetrahedral complex could be characterised as the set of lines meeting 4 given planes in points with a constant cross-ratio. The four planes are the singular planes of the complex. So we may now ask what the cross-ratio is for the complex of normals to the isogyre quadrics. In [18, p. 375] the canonical form for a tetrahedral complex is give as,

$$P_{24}P_{31} + \lambda P_{34}P_{12} = 0,$$

where λ is the cross-ratio of the complex. To this equation we may add any multiple of the Klein quadric, equation (1), without changing the complex. The equation derived above for this complex, (43), can be written in this notation as,

$$a^2P_{14}P_{23} + b^2P_{24}P_{31} + c^2P_{34}P_{12} = 0.$$

Hence, we can see that the cross-ratio of this tetrahedral complex is,

$$\lambda = \frac{c^2 - a^2}{b^2 - a^2} = \frac{k_1^2 - k_3^2}{k_1^2 - k_2^2}.$$

The last equality here is by virtue of equations (28)–(30) above.

Although the Painvin complex has Segre symbol [111111], it is a sub-type of the general type. For the Painvin complex the eigenvalues of the matrix Q_0Q are distinct but occur in positive/negative pairs, $\pm\lambda_i$. It is well known that for the Painvin complex the surface of singular points degenerates to Fresnel's wave surface. This surface is familiar from optics and the theory of elasticity, see for example [11, p. 125] or [9, p. 112]. However, different real forms of the wave surface can occur. For the complex of isogyre lines discussed above the different real forms depend on the value of the radius of gyration k with respect to the principal values of the inertia matrix.

For the complex of k-isogyre lines, the surface of singular points is defined by the equation,

$$\det\left(w^2(\mathbf{I} - k^2I_3) + P^TP\right) = 0.$$

The homogenising variable w, has been included here so that the surface will lie in $\mathbb{P}^3$.

Using coordinates centred on the centroid of the body and aligned with the principal axes diagonalises the inertia matrix $\mathbf{I}$ as usual. After some algebra, we obtain the equation,

$$\begin{aligned}(\alpha x^2 + \beta y^2 + \gamma z^2)(x^2 + y^2 + z^2) + \\ \alpha\beta(x^2 + y^2)w^2 + \beta\gamma(y^2 + z^2)w^2 + \alpha\gamma(x^2 + z^2)w^2 + \alpha\beta\gamma w^4 = 0,\end{aligned} \tag{47}$$

where,

$$\alpha = k_1^2 - k^2, \tag{48}$$

$$\beta = k_2^2 - k^2, \tag{49}$$

$$\gamma = k_3^2 - k^2. \tag{50}$$

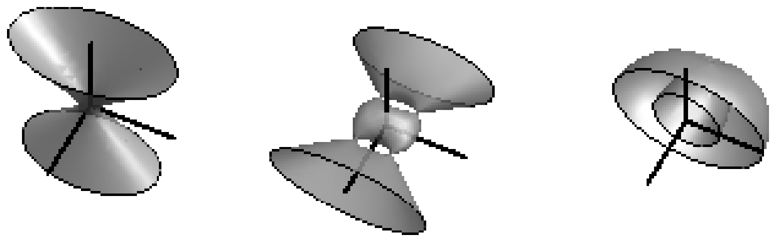

Fig. 5 Wave Surfaces for a Inertia matrix with Principal Inertias $k_1^2 = 2$, $k_2^2 = 3$, and $k_3^2 = 4$. The different values of k^2 are, left to right, $2.5, 3.5$ and 4.5. The surface on the right, $k^2 = 4.5$, has been cut-away so that the internal part of the surface can be seen.

For the standard Fresnel wave surface the coefficients of the w^2 terms are negative. In this case there are three possibilities depending on the size of k^2. Assuming, as usual, that $k_3^2 \geq k_2^2 \geq k_1^2$ then if $k^2 > k_3^2$ all α, β and γ will be negative. So in this case the coefficients of the w^4 and w^0 terms in (47) will be negative and the the coefficients of the w^2 terms will be positive and hence this will give the standard Fresnel wave surface. See the rightmost diagram in Fig. 5.

When $k_3^2 > k^2 > k_2^2$ then $\gamma > 0$ but α and β are negative. With this pattern of signs the surface of singular points is as shown in the middle diagram of Fig. 5.

In the case that $k_2^2 > k^2 > k_1^2$ only α is negative. The resulting surface is shown on the left of Fig. 5. Notice that in this case the surface has no real singularities, this is in contrast with the previous two cases which each have 4 real singularities.

Finally here, when $k_1^2 > k^2$ all of α, β and γ are positive and equation (47) has no real solutions.

5 Geometry of the Inertia Matrix

In this section we look again at some work of R.S. Ball. Although his influential book [2], was the foundation of 'screw theory', much of the book was taken up with investigating the problem of how an initially stationary rigid body would move when subjected to an impulsive wrench. This also leads to some new ideas, first presented here in this paper, concerning spatial percussion axes and how to diagonalise the mass matrix of a 2-joint robot.

5.1 Motion from Impulsive Wrenches

The equations of motion for a rigid-body can be written neatly in terms of twists and wrenches as,

$$\mathscr{W} = N\dot{\mathbf{s}} + \{\mathbf{s}, N\mathbf{s}\}, \tag{51}$$

see [16]. Here, $\mathscr{W}$ is the applied wrench, $\mathbf{s}$ the twist velocity and N the inertia matrix of the body, see eqs. (18) and (19). The operation $\{\cdot,\cdot\}$ is the co-adjoint action of a twist on a wrench. This equation combines Newton's equations with Euler's equations for rotational motion. Notice that from here the inertia matrix will contain the mass of the body, see below.

Now suppose a rigid body is subject to an instantaneous wrench $\mathscr{W}$ at time $t = 0$, and assume the body is initially at rest so that the Taylor expansion of the twist can be written,

$$\mathbf{s}(t) = t\mathbf{s}_1 + \frac{t^2}{2}\mathbf{s}_2 + \cdots \tag{52}$$

Substituting eq. (52) into the equation of motion (51) and setting $t = 0$ gives,

$$\mathscr{W} = N\mathbf{s}_1, \tag{53}$$

notice that it does not matter whether we think of the equation of motion as being written in a coordinate frame at rest with respect to the body or an inertial frame since we may assume that the two coincide at $t = 0$.

In Ball's language, if we apply an impulsive wrench $\mathscr{W}$ to the body it will begin to move by twisting about the screw $\mathbf{s}_1$.

5.2 Relations Between the Impulsive Wrench and Instantaneous Screw

An immediate consequence of the above relation is that we can easily see that:

Lemma 1 *To get the body moving with a translational motion, we must apply a pure force along a line through the centre of mass.*

Proof Let,

$$\mathbf{s}_1 = \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} \text{ and } N = \begin{pmatrix} \mathbf{I} & mC \\ mC^T & mI_3 \end{pmatrix},$$

where $\mathbf{v}$ is an arbitrary translation and C is the anti-symmetric matrix corresponding to the centroid $\mathbf{c}$ of the body. As mentioned above, the inertia matrix will contain the mass of the body. So unlike previously in this work, the diagonal form of the inertia matrix will be,

$$\mathbf{I} = \begin{pmatrix} mk_1^2 & 0 & 0 \\ 0 & mk_2^2 & 0 \\ 0 & 0 & mk_3^2 \end{pmatrix}, \tag{54}$$

where, as usual, k_i is the radius of gyration of the body about the ith principal axis and m is the mass of the body. Then substituting this in (53) gives,

$$\mathscr{W} = \begin{pmatrix} \mathbf{I} & mC \\ mC^T & mI_3 \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} m\mathbf{c} \times \mathbf{v} \\ m\mathbf{v} \end{pmatrix}.$$

□

Notice that the force is in the same direction as the translation.

For a general instantaneous screw motion, not a pure translation, we can write,

$$\mathbf{s}_1 = \begin{pmatrix} \omega \\ \mathbf{r} \times \omega + p\omega \end{pmatrix},$$

as usual, ω is the direction of the motion, $\mathbf{r}$ is a point on the axis of the screw and here p is the pitch of the screw. A general impulsive wrench can be written in partitioned form as,

$$\mathscr{W} = \begin{pmatrix} \mathbf{q} \times \mathbf{F} + h\mathbf{F} \\ \mathbf{F} \end{pmatrix},$$

where, $\mathbf{F}$ is the direction of the wrench, $\mathbf{q}$ is a point on the axis of the wrench and h is the pitch of the wrench. With the inertia matrix as above, the force component of the wrench is,

$$\mathbf{F} = m(\omega \times (\mathbf{c} - \mathbf{r}) + p\omega). \tag{55}$$

This allows us to answer the question: What happens if the body is subject to a pure torque? That is, how can $\mathbf{F} = \mathbf{0}$?

Lemma 2 *If the body is subject to a pure torque it will begin to rotate about a line through the centre of mass.*

Proof Since the two terms in equation (55) are perpendicular, they must vanish separately. That is, we must have that $p = 0$ and $\omega \times (\mathbf{c} - \mathbf{r}) = \mathbf{0}$. Furthermore, since $\mathbf{r}$ is a point on the axis of the screw, for ω to be parallel to $(\mathbf{c} - \mathbf{r})$ the centre of mass $\mathbf{c}$ must also lie on the screw axis. □

These results are well known, next we look at a pair of theorems given by Ball, [2, §301].

Theorem 3 *(Ball 1900) The perpendicular from the centre of gravity on any instantaneous screw is parallel to the shortest distance between that instantaneous screw and the corresponding impulsive screw.*

Theorem 4 *(Ball 1900) The perpendicular from the centre of gravity on any instantaneous screw is equal to the product of the pitch of that screw, and the tangent of the angle between it and the corresponding impulsive screw.*

In Ball's language the impulsive screw is the wrench $\mathscr{W}$, the direction of the axis for this screw is $\mathbf{F}$. The instantaneous screw is $\mathbf{s}_1$ with direction ω. The content of these theorems is illustrated in Fig. 6.

These theorems can be proved quite simply from the above. Taking the vector product of equation (55) with ω gives,

$$\omega \times \mathbf{F} = m\omega(\omega \cdot (\mathbf{c} - \mathbf{r})) - m|\omega|^2(\mathbf{c} - \mathbf{r}).$$

Now choose $\mathbf{r}$, the point on the axis of the instantaneous screw so that the vector $(\mathbf{c} - \mathbf{r})$ is perpendicular to ω; the direction of the instantaneous screw. The equation above will now read,

$$\omega \times \mathbf{F} = m|\omega|^2(\mathbf{r} - \mathbf{c}), \tag{56}$$

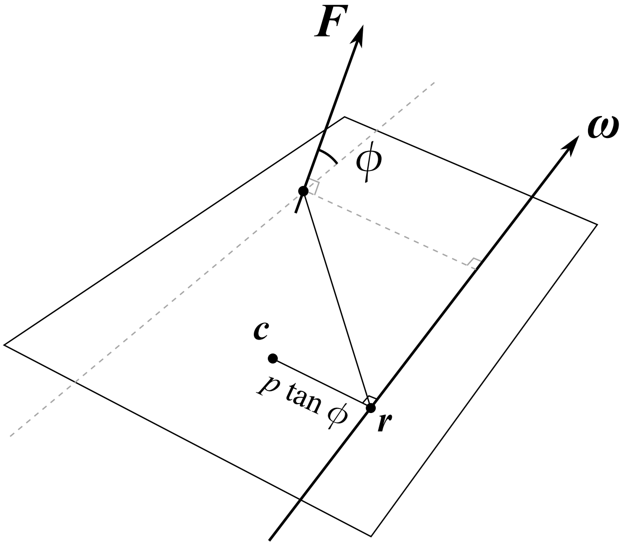

Fig. 6 The Two Theorems from Ball's Treatise.

proving Ball's theorem 3 above, since $(\mathbf{r}-\mathbf{c})$ is the perpendicular from the centre of gravity $\mathbf{c}$ to the axis of the instantaneous screw and $\omega \times \mathbf{F}$ is in the direction of the shortest distance between the instantaneous screw and the corresponding impulsive screw (or wrench).

Next take the modulus of the last equation,

$$|\omega \times \mathbf{F}| = |\mathbf{F}||\omega| \sin\phi = m|\omega|^2|\mathbf{c}-\mathbf{r}|,$$

where ϕ is the angle between the axes of the instantaneous screw and its corresponding impulsive screw, this can be chosen so that $0 \leq \phi < \pi$. Also we can take the scalar product of equation (55) with ω, this yields,

$$\omega \cdot \mathbf{F} = |\mathbf{F}||\omega| \cos\phi = mp|\omega|^2. \tag{57}$$

Dividing these last two equations gives the Ball's theorem 4 above:

$$\tan\phi = \frac{|\mathbf{c}-\mathbf{r}|}{p}, \tag{58}$$

or rather,

$$|\mathbf{c}-\mathbf{r}| = p\tan\phi. \tag{59}$$

Notice, that if the pitch of the motion vanishes $p = 0$, that is if the body begins to move with a pure rotation, then from equation (55) we can see that the axes of the impulsive wrench and instantaneous screw must be perpendicular. This means that the angle between the two is, $\phi = \frac{\pi}{2}$ and hence the right-hand-side of equation (59) is indeterminate.

5.3 Radius of Gyration

Using the same notation as in the previous section, we can give an expression for the radius of gyration about the axis of the instantaneous screw. To do this we look at the kinetic energy of the body,

$$E = \frac{1}{2}\mathscr{W}^T\mathbf{s} = \frac{1}{2}\mathbf{s}^T N\mathbf{s}.$$

Now if the pitch of the screw was zero $p = 0$, the leftmost expression in the above equation would give $(1/2)mk^2|\omega|^2$, where k is the radius of gyration about the axis of $\mathbf{s}$. When $p \neq= 0$ there are extra terms involving the pitch. These can be found by just considering the terms involving p in the rightmost expression above. In fact the terms involvng p only, vanish and only those containing p^2 are non-zero. Combining these results gives,

$$E = \frac{1}{2}m|\omega|^2(k^2 + p^2).$$

Next we can evaluate the kinetic energy using the expression involving the impulsive wrench, we get,

$$\mathscr{W}^T\mathbf{s} = (\mathbf{r} - \mathbf{q}) \cdot (\omega \times \mathbf{F}) + (p + h)\omega \cdot \mathbf{F}.$$

Equations (56) and (57) can be used to substitute for $\omega \times \mathbf{F}$ and $\omega \cdot \mathbf{F}$ to give,

$$\mathscr{W}^T\mathbf{s} = m|\omega|^2\big((\mathbf{r} - \mathbf{q}) \cdot (\mathbf{r} - \mathbf{c}) + p(p + h)\big).$$

Finally we compare the two expressions for the kinetic energy, cancelling common factors and rearranging gives the result,

$$k^2 = (\mathbf{r} - \mathbf{q}) \cdot (\mathbf{r} - \mathbf{c}) + ph \tag{60}$$

Notice that this expression does not depend on the mass of the body. It is also only dependant on the axis and pitch of the screw $\mathbf{s}$, multiplying ω by a non-zero scalar will not affect the result.

5.4 Generating the Same Twist from Two Inertially Distinct Bodies by applying the Same Wrench

Another problem addressed by Ball [2, §323], was to consider a pair of distinct rigid bodies with inertia matrices N_a and N_b and ask if it was possible that they could begin moving about the same instantaneous screw when subject to the same impulsive wrench. That is, suppose we subject the bodies to the same impulsive wrench $\mathscr{W}$ and ask when the starting twists satisfy $\mathbf{s}_b = \lambda\mathbf{s}_a = \lambda\mathbf{s}$ for some non-zero constant λ.

This can be expressed as,

$$\mathscr{W} = N_a\mathbf{s} = \lambda N_b\mathbf{s},$$

which implies that,

$$(N_a - \lambda N_b)\mathbf{s} = \mathbf{0}. \tag{61}$$

This equation has non-trivial solutions if and only if,

$$\det(N_a - \lambda N_b) = 0.$$

This is a generalised eigenvalue problem and hence we expect 6 solutions for λ and 6 corresponding values for $\mathbf{s}$. Moreover, since both N_a and N_b are positive definite symmetric matrices we expect the solutions to be real, see for example [12, §12.4]. It is important to note that we are assuming, for the moment, that the centres of mass for the two bodies are not coincident. We will return to the special case where the mass centres coincide at the end of this section.

One solution is easy to see. The two inertia matrices can be written in partitioned form,

$$N_a = \begin{pmatrix} \mathbf{I}_a & m_aC_a \\ m_aC_a^T & m_aI_3 \end{pmatrix}, \qquad N_b = \begin{pmatrix} \mathbf{I}_b & m_bC_b \\ m_bC_b^T & m_bI_3 \end{pmatrix}.$$

Now with $\lambda = m_a/m_b$ we get,

$$(N_a - \lambda N_b) = \begin{pmatrix} \mathbf{I}_a - \frac{m_a}{m_b}\mathbf{I}_b & m_a(C_a - C_b) \\ m_a(C_a - C_b)^T & 0 \end{pmatrix}.$$

which is clearly singular. The corresponding twist is given by,

$$\mathbf{s} = \begin{pmatrix} \mathbf{0} \\ \mathbf{c}_a - \mathbf{c}_b \end{pmatrix},$$

a pure translation parallel to the direction of the line of centres. The corresponding wrench will be,

$$\mathscr{W} = m_a \begin{pmatrix} \mathbf{c}_b \times \mathbf{c}_a \\ \mathbf{c}_a - \mathbf{c}_b \end{pmatrix},$$

that is a pure force along the line joining the mass centres of the two bodies.

From Theorem 3 above, the centre of gravity of each body will lie in the plane determined by the axis of the instantaneous screw, $\mathbf{s}$, and the line perpendicular to this axis and the axis of the impulsive wrench $\mathscr{W}$. From Theorem 4, if the pitch of the instantaneous wrench is non-zero $p \neq 0$, then the two centres of mass will be at the same distance from the axis of the instantaneous screw and hence will lie on a line parallel to the axis of $\mathbf{s}$.

Now consider the radius of gyration about the axis of $\mathbf{s}$. According to equation (60) we have for the first body,

$$k_a^2 = (\mathbf{r}_a - \mathbf{q}) \cdot (\mathbf{r}_a - \mathbf{c}_a) + ph,$$

and

$$k_b^2 = (\mathbf{r}_b - \mathbf{q}) \cdot (\mathbf{r}_b - \mathbf{c}_b) + ph,$$

for the second body. Now the vector expression on the right of these equations are the same, see Fig. 7, hence $k_a^2 = k_b^2$. That is the radius of gyration of the two bodies about $\mathbf{s}$ must be the same. This line of reasoning led Ball ([2, §323]) to conclude that, in general, there can be no other solutions, since in general the radius of gyration of two bodies about a given screw will be different. However, thinking of the problem as a generalised eigenvalue problem we have already observed that

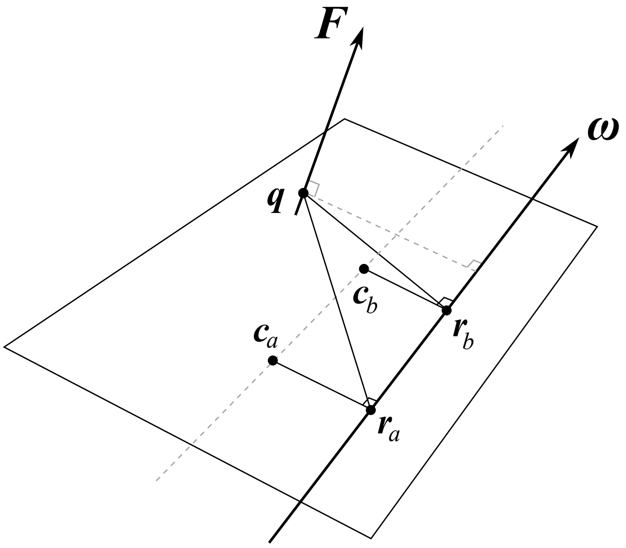

Fig. 7 Two bodies respond with the same instantaneous screw motion when subjected to the same impulsive wrench. The two centres of mass $\mathbf{c}_a$ and $\mathbf{c}_b$ lie on a line parallel to the axis of the screw and the points $\mathbf{r}_a$ and $\mathbf{r}_b$ are chosen so that both $(\mathbf{r}_a - \mathbf{c}_a)$ and $(\mathbf{r}_b - \mathbf{c}_b)$ are parallel to the common perpendicular between the screw and wrench. From the diagram it is clear that the projection of $(\mathbf{r}_a - \mathbf{q})$ and $(\mathbf{r}_b - \mathbf{q})$ onto the common perpendicular between the screw and wrench are the same. Hence, $(\mathbf{r}_a - \mathbf{q}) \cdot (\mathbf{r}_a - \mathbf{c}_a) = (\mathbf{r}_b - \mathbf{q}) \cdot (\mathbf{r}_b - \mathbf{c}_b)$.

there must be 6 real solutions in contradiction to Ball's conclusion. The resolution of this seeming paradox is to observe that Ball tacitly assumed that such screws would have non-zero pitch. We can see now that Ball's reasoning shows that in general we must have $p = 0$ for the other five solutions.

Assuming that the eigenscrews have zero pitch we can say a little about the possible eigenvalues. Suppose $\mathbf{s}$ is an eigenscrew with eigenvalue λ. Now take the equation defining the eigen-problem, (61), and pair it with $\mathbf{s}$. This gives,

$$\mathbf{s}^T N_a \mathbf{s} - \lambda \mathbf{s} N_b \mathbf{s} = m_a k_a^2 |\omega|^2 - \lambda m_b k_b^2 |\omega|^2 = \mathbf{0}.$$

rearranging yields,

$$\lambda = \frac{m_a k_a^2}{m_b k_b^2}.$$

Standard theory of generalised eigen-problems tell us that if $\mathbf{s}_i$ and $\mathbf{s}_j$ are eigenscrews corresponding to different eigenvalues then both,

$$\mathbf{s}_i^T N_a \mathbf{s}_j = 0 \qquad \text{and} \qquad \mathbf{s}_i^T N_b \mathbf{s}_j = 0.$$

Next here we look at the special case studied by Ball. Above we showed that, if $p \neq 0$ then the radii of gyration about the screw axis for the two bodies must be the same. Ball gave a slightly more useful characterisation, he showed that the radii of gyration about the line joining the mass centres must be identical. Since the axis of the instantaneous screw is parallel to the line joining the centre's of mass of the two bodies this can easily be proved using the parallel axis theorem of inertias.

Now consider again the computation above for the eigenvalue resulting in equation (60). Repeating this derivation but now assuming that the radius of gyration about the axis of the eigenscrew is the same for both bodies, say k^2, gives,

$$\lambda = \frac{m_a(k^2+p^2)}{m_b(k^2+p^2)} = \frac{m_a}{m_b}.$$

This is the same as the eigenvalue for the pure force along the line of mass centres solution given at the top of this section.

Using this the eigenvalue equation (61), can be written as,

$$(m_bN_a - m_aN_b)\mathbf{s} =$$
$$\begin{pmatrix} m_b\mathbf{I}_a - m_a\mathbf{I}_b & m_am_b(C_a-C_b) \\ m_am_b(C_a-C_b)^T & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \mathbf{r}\times\omega + p\omega \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

The second row of this vector equation is easily solved by setting $\omega = (\mathbf{c}_a - \mathbf{c}_b)$. Substituting this into the first row gives,

$$(m_b\mathbf{I}_a - m_a\mathbf{I}_b)(\mathbf{c}_a-\mathbf{c}_b) + m_am_b(\mathbf{c}_a-\mathbf{c}_b)\times\big(\mathbf{r}\times(\mathbf{c}_a-\mathbf{c}_b)\big) = \mathbf{0}.$$

As usual, since $\mathbf{r}$ represents any point on the axis of the screw, we can assume that $\mathbf{r}$ is perpendicular to the direction of $\omega = (\mathbf{c}_a - \mathbf{c}_b)$. The above equation can be rearranged to give,

$$\mathbf{r} = \frac{1}{m_am_b|\mathbf{c}_a-\mathbf{c}_b|^2}(m_a\mathbf{I}_b - m_b\mathbf{I}_a)(\mathbf{c}_a-\mathbf{c}_b).$$

Notice that the pitch of the screw p is not fixed by these equations. Hence, we get a one-parameter family of solutions, the eigenscrews can be of any pitch about the line through the point $\mathbf{r}$ in the direction $\omega = (\mathbf{c}_a - \mathbf{c}_b)$. This even extends to include the infinite pitch eigenscrew found above, the translation in the direction $\omega = (\mathbf{c}_a - \mathbf{c}_b)$. The corresponding eigenwrenches form a line in wrench space, hence their axes comprise a cylindroid.

Finally here, we return to the special case where the mass centres of the two bodies coincide. In this case we can choose to take the origin of our coodinates at the common centre of mass, the equation for the eigenscrews will be,

$$(N_a - \lambda N_b)\mathbf{s} = \begin{pmatrix} \mathbf{I}_a - \lambda\mathbf{I}_b & 0 \\ 0 & (m_a - \lambda m_b)I_3 \end{pmatrix} \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Clearly, if $\lambda = (m_a/m_b)$ then any infinite pitch screw will be a solution, $\mathbf{s}^T = (\mathbf{0}^T, \mathbf{v}^T)$. In general there will be three more solutions, pitch zero screws, with axes passing through the common centre of mass in the direction of the generalised eigenvector solutions to,

$$(\mathbf{I}_a - \lambda\mathbf{I}_b)\omega = \mathbf{0}.$$

5.5 Spatial Axes of Percussion

In the dynamics of lamina or two dimensional bodies, the centre of percussion is a point where an impulsive force produces only rotational motion. In this section we seek to generalise this concept to spatial motion. Specifically we seek an impulsive pure force which, when applied to a rigid body, causes an instantaneous motion which is a pure rotation about some line. This line will be referred here as the spatial axis of percussion, or simply the axis of percussion.

Again we begin with equation (53): $\mathscr{W} = N\mathbf{s}_1$. Now assuming that the motion is a pure rotation implies that the pitch p in equation (55) is zero and hence, $\mathbf{F}\cdot\omega = 0$. That is, the line of action of the force must be perpendicular to the rotation axis of the motion produced. Since the impulsive wrench $\mathscr{W}$ is a pure force it must satisfy $\mathscr{W}^T Q_0 \mathscr{W} = 0$. Substituting for $\mathscr{W}$ using equation (53) we get a quadratic equation for the lines which can be the axes for such motions,

$$(N\mathbf{s}_1)^T Q_0 N\mathbf{s}_1 = \\ \mathbf{s}_1{}^T(N^T Q_0 N)\mathbf{s}_1 = (\omega^T, \mathbf{v}^T)\begin{pmatrix} C\mathbf{I}-\mathbf{I}C & \mathbf{I}+mC^2 \\ \mathbf{I}+mC^2 & 0 \end{pmatrix}\begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} = 0. \quad (62)$$

Using coordinates with origin located at the body's centroid ($C = 0$) and axes along the principal axes of inertia ($\mathbf{I}$ diagonal), the symmetric matrix of the complex becomes,

$$\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$$

where in these coordinates, $\mathbf{I}$ is the 3×3 diagonal matrix as given in equation (54) above.

Notice that the lines determined by this equation are the same as the lines determined by the quadratic equation (43) above. From equation (25) we have that,

$$\begin{pmatrix} 0 & \Xi \\ \Xi & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} + \sigma^2\begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}.$$

The matrix on the extreme right of this equation is simply a multiple of the Klein form, see equation (3) above. All lines satisfy $\mathbf{s}^T Q_0 \mathbf{s} = 0$, hence this term can be ignored, see section 4.2. The difference between the two equations for the quadratic line complex is simply a change of basis in the pencil defining the complex. Hence we have the result that the percussion axes, as defined above, are exactly the normals to the isogyre quadrics.

The axes of the wrenches, pure forces, which produce these pure rotations also lie on a tetrahedral line complex. In the standard coordinates with the origin at the body's centroid, the equation of the complex is,

$$\mathscr{W}^T\begin{pmatrix} 0 & \mathbf{I}^{-1} \\ \mathbf{I}^{-1} & 0 \end{pmatrix}\mathscr{W} = 0,$$

showing the form of the 6×6 symmetric matrix determining this tetrahedral line complex.

The considerations above prompt the another question. Suppose we subject the body to an impulsive wrench with pitch h, how can this cause the body to begin to move with a pure rotation?

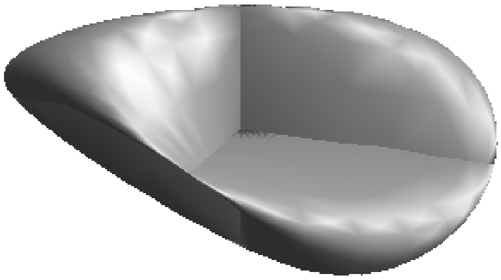

Fig. 8 Steiner surface, the singularity surface of the [222] line complex of axes the body can initially rotate about when subject to a pitch 1 wrench. The Principal Inertias of the body are again $k_1^2 = 4$, $k_2^2 = 3$, and $k_3^2 = 2$. Note that the line singularities of the Steiner surface are aligned with the principal axes of the body and the pinch point is located at the body's centroid.

The possible rotation axes $\mathbf{s}_1$, will satisfy the equation,

$$\mathbf{s}_1^T(N^T Q_0 N)\mathbf{s}_1 - 2h\mathbf{s}_1^T(N^T Q'_\infty N)\mathbf{s}_1 = 0,$$

where Q'_∞ is the 6×6 matrix,

$$Q'_\infty = \begin{pmatrix} 0 & 0 \\ 0 & I_3 \end{pmatrix}.$$

This defines another quadratic line complex. The 6×6 symmetric matrix for this complex is,

$$\begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & -2hI_3 \end{pmatrix},$$

where the standard coordinates, aligned with the principal axes and with origin at the body's centroid have been used. This quadratic line complex has Segre symbol [222], it does not seem to have any special name.

The equation for the surface of singularities can be computed as in section 4.5 above. The result is,

$$\alpha x^2y^2 + \beta x^2z^2 + \gamma y^2z^2 + \delta xyz = 0,$$

where,

$$\alpha = h(k_1^2 - k_2^2)^2, \tag{63}$$
$$\beta = h(k_1^2 - k_3^2)^2, \tag{64}$$
$$\gamma = h(k_2^2 - k_3^2)^2, \tag{65}$$
$$\delta = (k_1^2 - k_2^2)(k_1^2 - k_3^2)(k_2^2 - k_3^2). \tag{66}$$

This is the equation of a Steiner surface sometimes called the Roman surface or Steiner's roman surface, see Fig. 8.

Finally in this section, we look at a rather precise question. Is it possible to find an impulsive wrench $\mathscr{W}$ which when applied to the body causes it to begin to rotate about the screw $\mathbf{s}_1$ but where the axes of $\mathscr{W}$ and $\mathbf{s}_1$ meet?

Lemma 3 *No impulsive wrench can cause an instantaneous pure rotation about an axis which meets the axis of the wrench.*

Proof Assume that,

$$\mathbf{s}_1 = \begin{pmatrix} \omega \\ \mathbf{r} \times \omega \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & mI_3 \end{pmatrix},$$

where $\mathbf{r}$, the position vector of a point on the axis of $\mathbf{s}_1$ is perpendicular to ω. Now the impulsive wrench can be written,

$$\mathscr{W} = m \begin{pmatrix} \lambda\omega + \mu\mathbf{r} + h\mathbf{r} \times \omega \\ \mathbf{r} \times \omega \end{pmatrix}.$$

Here h is the pitch of the wrench and λ and μ are constants. Now the axis of the wrench is obtained by subtracting the torque in the same direction as the force, a pure force with the same axis as the original wrench is thus,

$$\mathscr{W}' = m \begin{pmatrix} \lambda\omega + \mu\mathbf{r} \\ \mathbf{r} \times \omega \end{pmatrix},$$

notice that ω and $\mathbf{r}$ are both perpendicular to the direction of the force and perpendicular to each other. The wrench and screw will meet if

$$\mathbf{s}_1^T \mathscr{W}' = 0.$$

That is if $m(\mathbf{r} \times \omega) \cdot (\mathbf{r} \times \omega) + \lambda|\omega|^2 = 0$. Rearranging this gives,

$$\lambda = -m|\mathbf{r}|^2,$$

which shows that λ must be negative, or more precisely, non-positive. Now the moment of the wrench $\mathscr{W}$, is given by,

$$\mathbf{I}\omega = \lambda\omega + \mu\mathbf{r} + hm\mathbf{r} \times \omega.$$

Taking the scalar product with ω gives,

$$\lambda|\omega|^2 = \omega^T \mathbf{I}\omega.$$

Since the inertia matrix is positive-definite this shows that λ must be strictly positive, hence there can be no solution for λ and thus no solution for $\mathbf{s}_1^T \mathscr{W}' = 0$. □

5.6 Diagonalising the Mass Matrix of a 2R Robot and Conjugate Screws of Inertia

In [2] Ball defines conjugate screws of inertia. Given a body with 6×6 inertia matrix N, see eqs. (18) and (19), if we subject the body to an impulsive wrench $\mathscr{W}_1$ then it will begin to move with twist $\mathbf{s}_1$. Now suppose that the wrench does no work on the twist $\mathbf{s}_2$. That is $\mathscr{W}_1^T \mathbf{s}_2 = 0$. Now the twists $\mathbf{s}_1$ and $\mathbf{s}_2$ are called conjugate twists of inertia. The amplitude of the twists is clearly irrelevant hence this a property of the screws. If two twists are conjugate in this way they satisfy,

$$\mathbf{s}_1^T N \mathbf{s}_2 = 0.$$

This shows clearly that if $\mathbf{s}_1$ is conjugate to $\mathbf{s}_2$ then $\mathbf{s}_2$ is conjugate to $\mathbf{s}_1$—the relation is symmetric. Notice that at the end of the previous section it was shown that the six eigenscrews found must be mutually conjugate with respect to the inertia matrices of both bodies.

Also notice that the set of lines, (pitch zero screws), conjugate to a given screw $\mathbf{s}_2$ say, form a linear line complex, see section 4.1 above. The pitch and central axis of this complex are not necessarily the same as the pitch and axis of $\mathbf{s}_2$. In fact this gives another generalised eigen-problem, the condition for the wrench $N\mathbf{s}$ to have the same pitch and axis as $\mathbf{s}$ itself is simply,

$$N\mathbf{s} = \lambda Q_0 \mathbf{s},$$

where λ is some constant. The six-solutions to this problem are known as the principal screws of inertia. Using the standard coordinates with the origin at the body's centroid and axes aligned with the principal axes of the 3×3 inertia matrix $\mathbf{I}$ it is straightforward to work out that the possible eigenvalues are $\lambda_i^+ = mk_i$ and $\lambda_i^- = -mk_i$ where $i = 1, 2, 3$. The 6 principal screws of inertia come in pairs aligned with the principal axes and all passing through the centroid of the body, the pitches of these screws are given by $\pm k_i$, $i = 1, 2, 3$.

From the general theory of eigenvectors we can see that these principal screws of inertia are mutually conjugate and also mutually reciprocal.

Now consider a two joint robot arm. Suppose we want to design the second link in such a way that the mass matrix for the robot is diagonal, this would greatly simplify the control problem for the machine.

In [16, §13.6] it was shown that the mass matrix of such an arrangement is given by the 2×2 matrix,

$$\begin{pmatrix} \mathbf{s}_1^T (N_1 + N_2) \mathbf{s}_1 & \mathbf{s}_1^T N_2 \mathbf{s}_2 \\ \mathbf{s}_2^T N_2 \mathbf{s}_1 & \mathbf{s}_2^T N_2 \mathbf{s}_2 \end{pmatrix},$$

where $\mathbf{s}_1, \mathbf{s}_2$ are the joint screws associated to the 2 joints of the robot and N_1, N_2 are the 6×6 inertia matrices of the two link of the machine. Clearly all that is required to diagonalise this matrix is to have $\mathbf{s}_1^T N_2 \mathbf{s}_2 = 0$. However, this relation must hold for every position of the joints. If the joints are both revolute joints then as joint 2 rotates we can use Rodrigues' formula to write,

$$\mathbf{s}_1^T \left(I_6 + \sin\theta_2 S_2 + (1 - \cos\theta_2) S_2^2 \right)^T N_2 \mathbf{s}_2 = 0. \qquad (67)$$

Here the matrix S_2 is the adjoint representation of the twist $\mathbf{s}_2$,

$$S_2 = \mathrm{ad}(\mathbf{s}_2) = \begin{pmatrix} \Omega_2 & 0 \\ V_2 & \Omega_2 \end{pmatrix},$$

as usual, Ω_2 and V_2 are the 3×3 matrices corresponding to the vectors ω_2 and $\mathbf{v}_2$ respectively. Notice, this formula is only valid because $\mathbf{s}_2$ is a pure rotation, that is $\omega_2 \cdot \mathbf{v}_2 = 0$. Since the equation above must hold for all values of the rotation angle θ_2, we have that the line $\mathbf{s}_1$ must lie in the three linear complexes defined by,

$$\mathbf{s}_1^T N_2 \mathbf{s}_2 = 0, \qquad \mathbf{s}_1^T S_2^T N_2 \mathbf{s}_2 = 0 \quad \text{and} \quad \mathbf{s}_1^T (S_2^T)^2 N_2 \mathbf{s}_2 = 0.$$

According to [11, Art. 58], three general linear complexes intersect in a regulus, however, these three complexes are not in general position.

As usual we will work in a particular coordinate system, but this time it will be convenient to choose the coordinate system so that the axis of the second joint is aligned with the z-axis of the coordinates and passes through the origin, so that,

$$\mathbf{s}_2 = \begin{pmatrix} \mathbf{k} \\ \mathbf{0} \end{pmatrix}.$$

The unit vectors in the x, y and z-directions will be written $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ here.

From section 5.1 above we know that the direction of the wrench $N_2\mathbf{s}_2$ will be perpendicular to the direction of $\mathbf{s}_2$, the coordinate frame can be further specified by choosing the y-direction to be aligned with the direction of the wrench $N_2\mathbf{s}_2$ and choosing the origin to be located where the plane containing the axis of $N_2\mathbf{s}_2$ and perpendicular to $\mathbf{s}_2$, meets the axis of $\mathbf{s}_2$. In these coordinates the wrench can be written,

$$N_2 \mathbf{s}_2 = m \begin{pmatrix} h\mathbf{j} + d\mathbf{k} \\ \mathbf{j} \end{pmatrix},$$

where d is the perpendicular distance between the axes of the screw $\mathbf{s}_2$ and the wrench $N_2\mathbf{s}_2$ and h is the pitch of the wrench. Notice that this assumes that the wrench has finite pitch, the case where the wrench has infinite pitch will be dealt with latter.

The coadjoint action of $\mathbf{s}_2$ on the wrench then gives,

$$S_2^T N_2 \mathbf{s}_2 = m \begin{pmatrix} h\mathbf{i} \\ \mathbf{i} \end{pmatrix} \quad \text{and} \quad (S_2^2)^T N_2 \mathbf{s}_2 = m \begin{pmatrix} h\mathbf{j} \\ \mathbf{j} \end{pmatrix}.$$

Since equation (67) must be true for all values of θ_2, we may choose the position of $\mathbf{s}_1$ to be anywhere in a circle (regulus) about $\mathbf{s}_2$. In particular, a convenient choice is to take $\mathbf{s}_1$ such that the perpendicular distance between $\mathbf{s}_1$ and $\mathbf{s}_2$ lies in the x-direction. However $\mathbf{s}_1$ may not lie in the same plane as the origin and the wrench. So a point on the axis will be written, $x\mathbf{i} + z\mathbf{k}$, this implies that the direction of $\mathbf{s}_1$ has no component in the x-direction so this screw can be written as,

$$\mathbf{s}_1 = \begin{pmatrix} \sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{k} \\ -z\sin\phi\,\mathbf{i} - x\cos\phi\,\mathbf{j} + x\sin\phi\,\mathbf{k} \end{pmatrix}$$

The equations for the three linear complexes then become,

$$0 = h\sin\phi + d\cos\phi - x\cos\phi, \quad (68)$$
$$0 = z\sin\phi. \quad (69)$$
$$0 = h\sin\phi - x\cos\phi. \quad (70)$$

Subtracting, (70) from (68) gives $0 = d\cos\phi$. However, by Lemma 3, the axes of $\mathbf{s}_2$ and $N_2\mathbf{s}_2$ cannot meet hence d cannot be zero. This implies that $\cos\phi = 0$ that is $\phi = \frac{\pi}{2}$ and the axes of $\mathbf{s}_1$ and $\mathbf{s}_2$ must be perpendicular. The other equations can now be satisfied if $z = 0$ and $h = 0$. If $h = 0$ then the wrench $N_2\mathbf{s}_2$ is a pure force and hence $\mathbf{s}_2$ is a percussion axis as described in section 5.5 above. The requirement that $z = 0$ implies that $\mathbf{s}_1$ must lie in the plane perpendicular to $\mathbf{s}_2$ and containing $N_2\mathbf{s}_2$. The quantity x is not constrained and hence there is a two-parameter family of possible lines $\mathbf{s}_1$.

Next we return to the special case where the wrench $N_2\mathbf{s}_2$, has infinite pitch. This occurs when the centre of mass of the second link lies on the axis of the second joint $\mathbf{s}_2$. In this case the wrench can be written as,

$$N_2\mathbf{s}_2 = m\begin{pmatrix} \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \\ \mathbf{0} \end{pmatrix}.$$

The coadjoint action of $\mathbf{s}_2$ then gives,

$$S_2^T N_2\mathbf{s}_2 = m\begin{pmatrix} \beta\mathbf{i} - \alpha\mathbf{j} \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad (S_2^2)^T N_2\mathbf{s}_2 = m\begin{pmatrix} \alpha\mathbf{i} + \beta\mathbf{j} \\ \mathbf{0} \end{pmatrix}.$$

The three equations to be solves are then,

$$0 = \beta\sin\phi + \gamma\cos\phi,$$
$$0 = -\alpha\sin\phi.$$
$$0 = \beta\sin\phi.$$

There are two possible solutions to this system of equations, first we could have $\alpha = \beta = \cos\phi = 0$. Notice that we can't have all three coefficients α, β and γ disappearing, since the inertia matrix is positive definite. Now when $\alpha = \beta = 0$ this means that the joint $\mathbf{s}_2$ is aligned with one of the principal directions of the inertia matrix. The requirement that $\cos\phi = 0$ means again that the joint $\mathbf{s}_1$ must be perpendicular to $\mathbf{s}_2$. However, for this solution there is no restriction on the plane that $\mathbf{s}_1$ must lie in.

The final possibility is that $\sin\phi = 0$ and $\gamma = 0$. This cannot happen because, $\gamma = \mathbf{s}_2^T N_2\mathbf{s}_2$ and since the inertia matrix is positive definite, this cannot vanish.

These results can be summarised as a theorem:

Theorem 5 *The mass matrix of a 2R robot can be diagonalised if and only if one the following conditions hold.*

– *The second joint is the axis of a principal screw of inertia and the first joint is perpendicular to the second.*

– *The second joint is a percussion axis of the second link, that is, a normal to an isogyre quadric determined by the second link, and the first joint axis must be located in the plane perpendicular to the second joint and containing the axis of the wrench $N_2\mathbf{s}_2$.*

Notice that in either case the two joint axes must be perpendicular. This is a rather more detailed result than that given in [16, §13.6.1], moreover it is clear that this gives a complete solution to the original problem—these are the only possibilities that can occur.

6 Conclusions

It has only been possible here to look at a small part of the classical literature in the area of Synthetic Mechanics. However, we hope that we have demonstrated that these methods have much to offer, not only in terms of exquisite geometry, but also for practical design of mechanisms. For example, it is clear that the idea of percussion axes is relevant to mechanism balancing.

Many further areas remain to be explored. Ball's treatise contains a lot of material on a rigid-body suspended by an arbitrary system of springs. The study of small vibrations in such systems are ideally suited to the methods presented here.

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