# Centrodes and Lie algebra

J.M. Selig

Faculty of Business, Computing and Info. Management.

London South Bank University, London SE1 0AA, U.K.

# Introduction

Revisit classical subject using Lie group techniques.

- Neater derivation of standard results.
- Methods can be generalised to higher dimensions.

Hope to convince you that the Lie algebra velocity is more fundamental than the centrode curves.

# Lie Group of Rigid Planar Motions

Write group elements as $3 \times 3$ matrices,

$$\begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}$$

# Transformation of Points

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

in partitioned form,

$$\begin{pmatrix} \mathbf{p}' \\ 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix}$$

# Rotation about an Arbitrary Point

Rotation about a point $\mathbf{q}$ is given by the conjugation,

$$\begin{pmatrix} I_2 & \mathbf{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & \mathbf{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_2 & -\mathbf{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & (I - R)\mathbf{q} \\ 0 & 1 \end{pmatrix}$$

Matrix $I_2$ is $2 \times 2$ identity.
All rigid planar motions are rotations about a fixed point or pure translations.

# Active vs. Passive Transformation

Everything above in active view—transformations move points. Passive view—transformations are changes of coordinates.
Assume $M$ an active transformation that transforms coordinate frame $\Sigma$ to new frame ${}^n\Sigma$.
Then a point with coordinates $\mathbf{p}$ in $\Sigma$ has coordinates ${}^n\mathbf{p}$ in ${}^n\Sigma$ given by,

$$\begin{pmatrix} {}^n\mathbf{p} \\ 1 \end{pmatrix} = M^{-1}\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R^T & -R^T\mathbf{t} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

# Lie Algebra

To find Lie algebra elements differentiate 1-parameter subgroups at the identity. Rotations about $\mathbf{q}$,

$$\frac{d}{d\theta}\begin{pmatrix}\cos\theta & -\sin\theta & (1-\cos\theta)q_x+\sin\theta q_y\\ \sin\theta & \cos\theta & -\sin\theta q_x+(1-\cos\theta)q_y\\ 0 & 0 & 1\end{pmatrix}\Bigg|_{\theta=0}$$

$$=\begin{pmatrix}0 & -1 & q_y\\ 1 & 0 & -q_x\\ 0 & 0 & 0\end{pmatrix}.$$

Write $E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, Lie algebra element corresponding to a rotation about $\mathbf{q}$ is,

$$L = \begin{pmatrix} -E & E\mathbf{q} \\ 0 & 0 \end{pmatrix}$$

Matrix $L$ satisfies $L^3 + L = 0$ so satisfy familiar Rodrigues formula,

$$e^{\theta L} = I_3 + \sin\theta L + (1 - \cos\theta)L^2$$

# Derivatives

If $\theta = \theta(t)$ and $L = \begin{pmatrix} -E & E\mathbf{r}(t) \\ 0 & 0 \end{pmatrix}$ then the derivative of the exponential satisfies,

$$\frac{d}{dt}(e^{\theta L}) = L_d e^{\theta L}$$

where

$$L_d = \dot{\theta}\begin{pmatrix} -E & E\mathbf{r} \\ 0 & 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & E\dot{\mathbf{r}} \\ 0 & 0 \end{pmatrix} + (1-\cos\theta)\begin{pmatrix} 0 & \dot{\mathbf{r}} \\ 0 & 0 \end{pmatrix}$$

# The Pole of a Motion

Rigid motion represented by a curve in the group $M(t) = e^{\theta L}$. The pole of the motion is the point that is instantaneously at rest.

If $\begin{pmatrix} \mathbf{p}(t) \\ 1 \end{pmatrix} = M(t) \begin{pmatrix} \mathbf{p}_0 \\ 1 \end{pmatrix}$ then the velocity is

$$\begin{pmatrix} \dot{\mathbf{p}} \\ 0 \end{pmatrix} = L_d e^{\theta L} \begin{pmatrix} \mathbf{p}_0 \\ 1 \end{pmatrix} = L_d \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix}$$

The pole satisfies,

$$L_d \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \mathbf{0}$$

# The Fixed Centrode

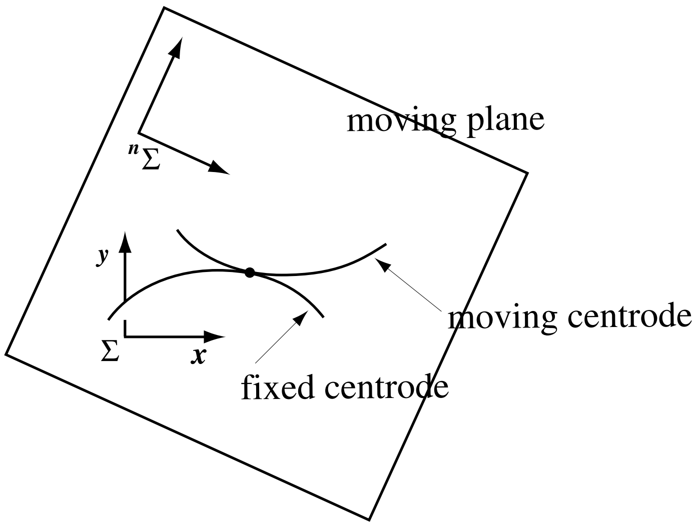
Write $L_d = \omega \begin{pmatrix} -E & E\mathbf{c}_f \\ 0 & 0 \end{pmatrix}$, pole then satisfies,

$$\omega(\mathbf{p} - \mathbf{c}_f) = \mathbf{0}$$

if $\omega \neq 0$, see that pole is $\mathbf{c}_f$. As $t$ varies the instantaneous poles $\mathbf{c}_f$ trace out the fixed centrode.
From above can see $\omega = \dot{\theta}$, the angular velocity of the motion and,

$$\mathbf{c}_f = \mathbf{r} + \frac{1}{\dot{\theta}} \begin{pmatrix} \sin\theta & \cos\theta - 1 \\ 1 - \cos\theta & \sin\theta \end{pmatrix} \dot{\mathbf{r}}$$

# The Moving Centrode




The moving centrode is the same pole point $\mathbf{p}$ but given in a co-ordinate frame in the moving plane. Given by the derivative in the moving frame.

$$\mathbf{c}_m = \mathbf{r} + \frac{1}{\dot{\theta}} \begin{pmatrix} \sin\theta & 1-\cos\theta \\ \cos\theta - 1 & \sin\theta \end{pmatrix} \dot{\mathbf{r}}$$

# Example - Cardan Motion

Motion can be written as product of exponentials,

$$M(t) = e^{-\sin\theta J} e^{\theta L}$$

where $L = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Derivative in the fixed frame is,

$$\begin{aligned} \left(\frac{d}{dt}M\right) M^{-1} &= -\dot{\theta}\cos\theta J + \dot{\theta} e^{-\sin\theta J} L e^{\sin\theta J} \\ &= \dot{\theta} \begin{pmatrix} 0 & -1 & -\sin\theta \\ 1 & 0 & -\cos\theta \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence fixed centrode is,

$$\mathbf{c}_f = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

Similarly, looking at the derivative in the moving frame $M^{-1}\left(\frac{d}{dt}M\right)$, gives the moving centrode,

$$\mathbf{c}_m = \begin{pmatrix} (1/2) + (1/2)\cos 2\theta \\ -(1/2)\sin 2\theta \end{pmatrix}$$

Two circles, as expected.

# Rolling Curves

Here look at motion generated by one curve rolling on another fixed curve. Fixed curve,

$$\mathbf{f}(t) = \begin{pmatrix} \mathbf{p}_f(t) \\ 1 \end{pmatrix}, \text{ and } \mathbf{m}(t) = \begin{pmatrix} \mathbf{p}_m(t) \\ 1 \end{pmatrix}$$

the moving curve in the moving frame. Curves meet at time $t$,

$$\mathbf{f}(t) = e^{\theta L}\mathbf{m}(t)$$

For rolling, tangents parallel at time $t$, so

$$(1+\gamma(t))\dot{\mathbf{f}}(t) = e^{\theta L}\dot{\mathbf{m}}(t)$$

where the scalar function $\gamma(t)$ measures the slip, (for no slipping $\gamma = 0$).
Differentiate contact condition,

$$\begin{aligned} \dot{\mathbf{f}}(t) &= L_d e^{\theta L}\mathbf{m}(t) + e^{\theta L}\dot{\mathbf{m}}(t) \\ &= L_d\mathbf{f}(t) + e^{\theta L}\dot{\mathbf{m}}(t) \\ &= L_d\mathbf{f}(t) + (1+\gamma(t))\dot{\mathbf{f}}(t) \end{aligned}$$

Can easily show that, $\omega\dot{\mathbf{p}}_f \cdot (\mathbf{p}_f - \mathbf{c}_f) = 0$ — pole lies on the normal to the fixed curve.
Further, for the fixed centrode,

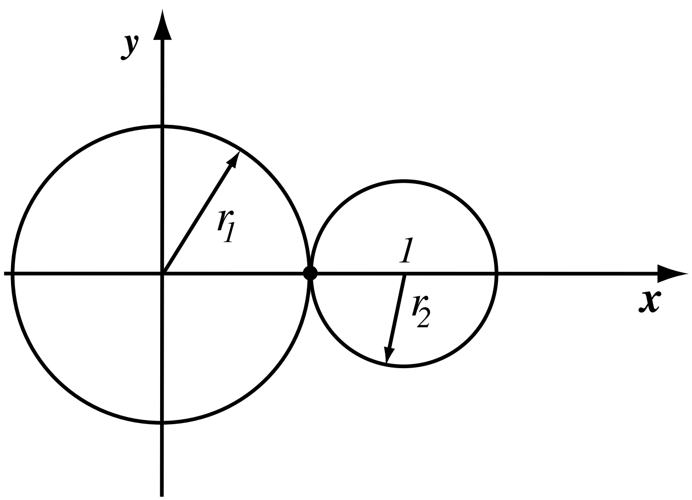
$$\mathbf{c}_f = \mathbf{p}_f + \frac{\gamma}{\omega}E\dot{\mathbf{p}}_f$$

and for the moving centrode,

$$\mathbf{c}_m = \mathbf{p}_m + \frac{\gamma}{\omega(1+\gamma)}E\dot{\mathbf{p}}_m$$

If $\gamma = 0$ $\mathbf{c}_f = \mathbf{p}_f$ and $\mathbf{c}_m = \mathbf{p}_m$.

# Example - Gear Teeth




An old but important problem. Think of motion of pinion in frame fixed in the larger gear. For simplicity $r_1 + r_2 = 1$ and for no slipping $\alpha r_1 = \beta r_2$ where $\alpha$ and $\beta$ are angular velocities of the gears.

Clearly the centrodes are given by the circles, for example fixed centrode is.

$$\mathbf{c}_f = \begin{pmatrix} r_1 \cos \alpha t \\ r_1 \sin \alpha t \end{pmatrix}$$

Assume fixed curve given by,

$$\mathbf{p}_f = \begin{pmatrix} r_1 \cos \alpha t + \alpha r_1 t \sin \alpha t \\ r_1 \sin \alpha t - \alpha r_1 t \cos \alpha t \end{pmatrix}$$

involute of the fixed centrode circle.

Velocity/tangent to this curve is,

$$\dot{\mathbf{p}}_f = \begin{pmatrix} \alpha^2 r_1 t \cos \alpha t \\ \alpha^2 r_1 t \sin \alpha t \end{pmatrix}$$

Verifies that $\mathbf{c}_f$ lies on normal to $\mathbf{p}_f$ – consistency condition.
From this can derive the moving curve,

$$\mathbf{p}_m = \begin{pmatrix} 1 - r_2 \cos \beta t - \beta r_2 t \sin \beta t \\ r_2 \sin \beta t - \beta r_2 t \cos \beta t \end{pmatrix}$$

also involute of a circle – of course!

Also the slip factor, in general,

$$\gamma(t) = (|\dot{\mathbf{p}}_m| - |\dot{\mathbf{p}}_f|)/|\dot{\mathbf{p}}_f|$$

here,

$$\gamma(t) = (r_1 - r_2)/r_1$$

a constant.

# Conclusions

- Centrodes may not exist if $\omega = 0$ (instantaneous translation), Lie algebra velocity $L_d$ always exists.
- Can be extended to other problems in planar kinematics, e.g. inflection circle, Burmester points etc.
- Should extend relatively easily to Spatial motion, motions of lines and so forth.