Extension of dependence properties to semi-copulas and applications to the mean-variance model

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Abstract

This paper deals with the construction of a semi-copula D, not necessarily exchangeable, whose "dependence" properties translate remarkable aspects of investors' behavior. To achieve this aim, we propose a new version of the standard mean-variance framework. For our purpose, a particular class of utility functions G has been introduced. The induced transformation of G is considered and the definition of semi-copula D hinges on the family of the indifference curves of G.

Keywords: Indifference curves; semi-copulas; dependence properties; comparison between risky assets.

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1 Introduction

The well-known notion of copula (see e.g. Joe, 1997, Nelsen, 1999 and McNeil et al. 2005) finds a very natural extension to the concept of *semi-copula*. The latter has arisen in the field of reliability for the description of the family of level curves of a *joint survival function* $\overline{F}$ of a vector of non-negative random variables (see, in particular, Bassan and Spizzichino, 2005, Durante and Spizzichino, 2010, Durante et al., 2010). In this frame it has been shown that some properties of stochastic dependence of copulas can be extended to the field of semi-copulas in a natural way. Such "dependence" properties are interesting in that they reveal to be equivalent to some remarkable *multivariate ageing* properties of $\overline{F}$. The same concept of semi-copula was also studied in details, from a more analytical viewpoint, in Durante and Sempi (2005) and in Durante et al. (2006). From a more general mathematical perspective semi-copulas can be seen as special cases of Aggregation Operators and are strictly related to Triangular Norms (see in particular Klement et al., 2000).

In the present paper we rather address to the celebrated mean-variance model. This topic is of course completely classic (Markowitz, 1952, 1956, 1959 and Roy, 1952; see also De Finetti, 1940 and Markowitz, 2006). As well-known, however, it is still of actual interest nowadays and we aim here to use the concept of semi-copula in order to analyze some special aspects of such a framework.

We consider, for the mean-variance plane, a modified version Υ, where any risky asset Y is represented by the pair (u, v), u being the inverted mean of Y and v its standard deviation.

For an investor I who takes a position in the market, we consider the utility function $G(u, v)$ and the class $\mathcal{L}_G$ of the indifference curves of G over the plane Υ.

We then introduce a semi-copula D by means of a suitable transformation of the utility G (see Formulas (7) and (13) below) and show that D is adapt to describe the class $\mathcal{L}_G$. As we shall see, D is suitably build up by determining, for any risky asset Y, the riskless asset H_Y that turns out to be equivalent (in terms of the utility G) to Y. From a technical point of view, a main difference with previous applications of semi-copulas is that we deal here with functions that, generally, are not permutation-invariant.

The main purpose of this paper is to highlight that, in the present frame, some "dependence" properties of D correspond to different aspects of the behavior of I. More in particular, we consider in details relevant "dependence" properties such as *independence, positive quadrant dependence, left tail dependence and totally positivity of order 2*. We then explore the economic meaning arising when we impose such properties to the semi-copula D. As we shall see, this procedure leads us to obtain some conditions, on the behavior of I, of actual economic relevance.

More precisely the paper is organized as follows. In Section 2 we introduce the necessary preliminaries, notation, and definitions. In particular we define the plane Υ and the semi-copula D. In Section 3 we recall the definitions of the "dependence" properties of interest and explore the economic meaning they achieve when they are extended to D. Section 4 is devoted to a short discussion and some final remarks of economic interest.

2 Preliminaries and notation

We consider a probability space $(\Omega, \mathcal{F}, P)$ over which all the random variables, that will be introduced throughout the paper, are defined. In the classical Markowitz one-period model (Markowitz, 1952), the stochastic returns of the assets are assumed to be fully described by their (positive) mean and variance. More precisely, Markowitz assumes that they obey a normal law. Therefore, we consider $X \sim N(0,1)$ and define any risky asset by means of a stochastic return of the form

$$Y = \sqrt{\mathbb{V}(Y)} \cdot X + \mathbb{E}(Y), \tag{1}$$

where $\mathbb{E}$ and $\mathbb{V}$ denote the usual expected value and variance operator and $\mathbb{E}(Y) > 0$. As $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$ change, the asset Y will be represented in the mean-variance plane.

In order to develop our discussion, it is convenient using the same scale for risk and expected value of the stochastic returns. For this reason, we replace variance with standard deviation and then consider the mean-standard deviation plane. Let us introduce the mean-standard deviation criterion, according to which Y_1 is preferred to Y_2 whenever $\mathbb{E}(Y_2) \geq \mathbb{E}(Y_1)$ and $\sqrt{\mathbb{V}(Y_2)} \leq \sqrt{\mathbb{V}(Y_1)}$, with at least one strict inequality. To pursue the aim of this research, we need to rethink the mean-standard deviation framework as follows:

Definition 2.1. *Consider a risky asset Y defined as in (1) and introduce two functionals u and v as follows:*

$$u(Y) = u_Y = \sqrt{\mathbb{V}(Y)} \in [0, +\infty); \qquad v(Y) = v_Y = (\mathbb{E}(Y))^{-1} \in (0, +\infty). \quad (2)$$

Moreover, extend the functional v (to the non-normal case) by setting $v_Y = 0$ when the expected return of the risky asset Y is $+\infty$.
We will denote by $\Upsilon \equiv [0, +\infty)^2$ the quadrant containing the random amounts Y's with coordinates given by $u_Y \in [0, +\infty)$ and $v_Y \in [0, +\infty)$.

The set Υ may be viewed as a new version of the classical mean-standard deviation plane, obtained by reverting the parameter related to the mean.
It is worth noting that the usual mean-standard deviation framework provides also information about the *feasibility* of a risky asset, requiring then the introduction of an *admissible region* $\Gamma \subset \Upsilon$. We will deal with this issue later on.
The points of Υ correspond to all the risky assets, in that any asset Y is assumed to be fully determined by the related values of u and v. This property of Υ can be viewed as a reinterpretation of the analogous feature of the mean-standard deviation plane, where the assets are assumed to be normal, and then fully determined by the values taken by mean and standard deviation. We will denote a risky asset Y by means of u and v as follows:

$$Y \equiv [u, v]. \quad (3)$$

The introduction of Υ in Definition 2.1 leads to an immediate reinterpretation of the mean-standard deviation criterion (Markowitz, 1952). In particular, the set Υ can be endowed with a partial order, that is the translation in this setting of the usual mean-standard deviation decision rule.

Definition 2.2. *Given $u_1, v_1, u_2, v_2 \in [0, +\infty)$ and two risky assets $Y_1 \equiv [u_1, v_1]$ and $Y_2 \equiv [u_2, v_2]$, we say that $Y_1 \succ Y_2$ if and only if*

$$\begin{cases} u_1 \leq u_2 \\ v_1 \leq v_2 \end{cases} \quad (4)$$

with at least one strict inequality.

Let us place the variable u on the axis of abscissas and the variable v on the axis of ordinates.

As already noted above, condition (4) in Definition 2.2 establishes a partial order among the points belonging to the plane Υ. When Y_1 and Y_2 are not comparable through $\succ$, then the assessment of a utility function $G : \Upsilon \to \mathbb{R}$, describing investors' preferences, is needed.

In general, given an asset $Y \equiv [u, v] \in \Upsilon$, the utility function G assigns a real number $G(u, v)$ to Y which describes the *level of satisfaction* in owning Y.

For conventional agreement, individuals with utility function G are assumed to prefer assets with higher values of $G(u, v)$.

The usual properties of the utility functions in the mean-standard deviation setting can be translated in our context as follows:

($G1$) Fixed $u \in [0, +\infty)$, then $G(u, v)$ is strictly decreasing w.r.t. v.

($G2$) Fixed $v \in [0, +\infty)$, then $G(u, v)$ is strictly decreasing w.r.t. u.

As required by the standard Markowitz theory, conditions ($G1$) and ($G2$) assure that G is consistent with the decision criterion in Definition 2.2. Indeed, such conditions imply that if $[u_1, v_1] \succ [u_2, v_2]$ then $G(u_1, v_1) > G(u_2, v_2)$.

Moreover, in order to meet financial reasonableness of the model, the preferences of the investor should not present jumps; rather they should vary with continuity w.r.t. both the inverted mean and the standard deviation of the available risky assets. Hence, we assume that the following condition holds:

($G3$) G is a continuous function of its arguments.

Consider now the family of the level curves of the utility function G:

$$\mathcal{L}_G = \left\{ l_G^{(\alpha)} \right\}_{\alpha \in \mathbb{R}}, \tag{5}$$

where

$$l_G^{(\alpha)} = \{(u, v) \in \Upsilon \mid G(u, v) = \alpha\}, \qquad \forall \, \alpha \in \mathbb{R}. \tag{6}$$

In agreement with the classical theory, the level curves of G are looked at as *indifference curves* and we say that the elements belonging to one and the same indifference curve are *G-equivalent*.

By means of the indifference curves, we can define a total order in Υ.

Definition 2.3. *Given $Y_1, Y_2 \in \Upsilon$, then $Y_1 \succ_C Y_2$ if and only if there exist $\alpha_1 > \alpha_2$ such that $Y_1 \in l_G^{(\alpha_1)}$ and $Y_2 \in l_G^{(\alpha_2)}$. The subscript C in symbol $\succ_C$ stands for Curve.*

Remark 2.1. *The orders $\succ$ and $\succ_C$ have analogous financial meaning. In fact, they define two different ways to identify which is the more profitable between two assets. We will then refer hereafter to profitability, in the sense of Definition 2.2 or in the sense of Definition 2.3. We notice however that $\succ$ is a partial order based on objective features of the assets, while $\succ_C$ is a total order grounded on a subjective perception of the investor.*

By taking into account conditions $(G1)$, $(G2)$, $(G3)$ of the utility function G, we can easily list the properties of the isoutility curves in our framework:

$(L1)$ When non-empty, $l_G^{(\alpha)}$ is a decreasing curve, for each $\alpha \in \mathbb{R}$.

$(L2)$ $l_G^{(\alpha_1)} \cap l_G^{(\alpha_2)} = \emptyset$, for $\alpha_1 \neq \alpha_2$.

$(L3)$ If $Y_1 \succ Y_2$, then $Y_1 \succ_C Y_2$. The inverse implication is not necessarily true.

We now carry out the analysis of the above setting by using a semi-copula approach. To this purpose, we project the points of Υ on the v-axis through the indifference curves. More formally, we define a function $h : \Upsilon \to [0, +\infty)$ as follows:

$$h(u, v) = \bar{v}, \tag{7}$$

where $\bar{v}$ is such that

$$G(u, v) = G(0, \bar{v}). \tag{8}$$

In order to simplify notation, we introduce the auxiliary function $\phi : [0, +\infty) \to \mathbb{R}$ as follows:

$$\phi(v) = G(0, v). \tag{9}$$

By definition, ϕ is strictly decreasing in $[0, +\infty)$ and continuous, and its inverse ϕ^{-1} is strictly decreasing as well.
A further technical requirement on G is needed to assure the well-posedness of the definition of h.

$(G4)$ $G([0, +\infty)^2) \subseteq \phi([0, +\infty))$.

Condition $(G4)$, namely that the codomain of G is contained in the codomain of ϕ, is needed to guarantee that each indifference curve intersects the v-axis. At this point, the definition of h is well-posed since, for each $(u, v) \in \Upsilon$, there exists a unique $\bar{v} > 0$ such that condition (8) is satisfied. Indeed, by (7), (8) and (9), we have:

$$h(u, v) = \phi^{-1}(G(u, v)), \qquad \forall u,\, v. \tag{10}$$

From an economic point of view, condition $(G4)$ states that levels of utility obtainable with risk-free assets spam the entire range of the values taken by utility function G. $(G4)$ can be seen as a sort of risk-aversion condition in that it is associated to utilities which can be maximized by only investing in risk-free assets.

We have:

Lemma 2.1. *Given $v \in [0, +\infty)$, then $h(u, v)$ is increasing w.r.t. u.*
Given $u \in [0, +\infty)$, then $h(u, v)$ is increasing w.r.t. v.

Proof. Since ϕ^{-1} is a decreasing function in $\mathbb{R}$ and G decreases with respect to its arguments as well, by formula (10) we obtain the thesis. □

Furthermore, condition $(G3)$ and definition of h imply that h is continuous w.r.t. its arguments as well.

Remark 2.2. *Consider $Y_1 = [u_1, v_1]$ and $Y_2 = [u_2, v_2]$. In view of $(G1)$ and $(G2)$, $Y_1 \succ Y_2$ implies that $h(u_1, v_1) < h(u_2, v_2)$. However, the condition $h(u_1, v_1) < h(u_2, v_2)$ is compatible with a non comparability between Y_1 and Y_2 and only excludes that $Y_2 \succ Y_1$.*

Some technical Lemmas are now needed, to show further properties of h.

Lemma 2.2. *We have $h(0, 0) = 0$.*

Proof. Suppose that there exists $\bar{v} > 0$ such that $h(0, 0) = \bar{v}$ and take $\tilde{v} < \bar{v}$. There exist $u, v > 0$ such that $h(u, v) = \tilde{v}$. Therefore, by Remark 2.2 we have that $[0, 0] \succ [u, v]$ is not true, and this is a contradiction (see Definition 2.2). □

Lemma 2.3. *One has:*

$$\lim_{v \to +\infty} h(u, v) = +\infty, \qquad \forall\, u \in [0, +\infty). \tag{11}$$

Proof. Let us consider two values $v, \bar{v} \in [0, +\infty)$. If some value $u \in [0, +\infty)$ exists such that $\phi^{-1}(G(u, v)) = \bar{v}$ then, by (8), we must have $v < \bar{v}$. Therefore, by continuity of $\phi^{-1}(G(u, v))$ w.r.t. v, we have:

$$\lim_{v \in [0, +\infty)} \phi^{-1}(G(u, v)) = \phi^{-1}\left(G\left(u, \lim_{v \in [0, +\infty)} v\right)\right) = +\infty,$$

and (11) holds. $\square$

Now, starting from the utility function G defined above, we define the function $\psi : [0, +\infty) \to [0, +\infty)$ as follows:

$$\psi(u) = h(u, 0). \tag{12}$$

The term $h(u, 0)$ represents a measure (in terms of expected return) of the *absolute* perception of the risk u by investor I. Such a measure is to be intended in the sense that -as an example- a large value of $h(u, 0)$ means that the presence of the risk u is responsible of a small expected return of the G-equivalent riskless bond, even in a situation of infinite expected return. By definition, the function ψ is strictly increasing in $[0, +\infty)$.

As mentioned in the Introduction, the concept of semi-copula generalizes the one of copula. To be self-contained, we recall here the definition of bivariate semi-copulas (see Bassan and Spizzichino, 2005; Durante and Sempi, 2005; Durante et al., 2006).

Definition 2.4. *A function $B : [0, 1]^2 \to [0, 1]$ is a bivariate semi-copula if and only if it satisfies the following conditions:*

(*a*) *$B(x, y)$ is increasing w.r.t. x, $\forall\, y \in [0, 1]$.*

(*b*) *$B(x, y)$ is increasing w.r.t. y, $\forall\, x \in [0, 1]$.*

(*c*) *$B(x, 1) = x$, $\forall\, x \in [0, 1]$.*

(*d*) *$B(1, y) = y$, $\forall\, y \in [0, 1]$.*

We can now introduce a function which plays a central role in our study.

Definition 2.5. *Let h be defined as in (7) and let us define $D : [0,1]^2 \to [0,1]$ as follows:*

$$\begin{cases} D(x,y) = \exp\{-h\left(\psi^{-1}(-log(x)), -log(y)\right)\}, & x > 0,\ y > 0; \\ D(x,0) = 0, & x \geq 0; \\ D(0,y) = 0, & y \geq 0. \end{cases} \tag{13}$$

Proposition 2.1. *The function D introduced in Definition 2.5 is a continuous semi-copula.*

Proof. We need to check if D satisfies $(a) - (d)$ of Definition 2.4.

(a) It is sufficient to check that $D(x,y)$ increases for $x \in (0,1]$, for each $y \in [0,1]$, being the exponential greater than 0.
Function $f(x) = -log(x)$ is decreasing in $(0,1]$. Moreover, ψ^{-1} increases, since ψ is increasing. $h(u,v)$ increases w.r.t. u. Then, $h\left(\psi^{-1}(-log(x)), -log(y)\right)$ decreases w.r.t. x, and so $\exp\{-h\left(\psi^{-1}(-log(x)), -log(y)\right)\}$ increases w.r.t. x. This gives the proof.

(b) The proof is analogous to that of case (a).

(c) By definition of ψ in (12) we have that

$$h(\psi^{-1}(-log(x)), 0) = -log(x). \tag{14}$$

Therefore:

$$D(x,1) = \exp\{-h\left(\psi^{-1}(-log(x)), 0\right)\} = \exp\{-(-log(x))\} = x.$$

(d) By Lemma 2.2 and definition of ψ in (12) we have $\psi^{-1}(0) = 0$.
Therefore:

$$D(1,y) = \exp\{-h\left(\psi^{-1}(0), -log(y)\right)\} =$$

$$= \exp\{-h\left(0, -log(y)\right)\} = \exp\{-(-log(y))\} = y.$$

□

We will refer to function D in (13) as the *mean-variance semi-copula*.

3 "Dependence" properties of semi-copulas and properties of utility functions

It is well-known that properties of stochastic dependence between pairs of random variables are suitably described by means of the notion of copula (see Nelsen, 1999 and Joe, 1997). In this respect, it is worth noting that some specific, well-established, dependence properties for bivariate copulas have been repeatedly considered in the statistical and financial literature (among the others, see Cherubini et al., 2004 and McNeil et al., 2005).

From a purely analytical point of view, some of these properties can be extended in a straightforward way, even to bivariate semi-copulas (see Bassan and Spizzichino, 2005; Spizzichino, 2010). Generally, the extension of dependence concepts from the field of copulas to the field of semi-copulas is only formal, and semi-copulas do not necessarily play a role in stochastic dependence analysis. As the main purpose of this paper, in this section we show that such an extension has rather an economic meaning when referred to our semi-copula D. Consider the utility function $G : \Upsilon \to \mathbb{R}$, that describes the preferences of investor I, and the corresponding mean-variance semi-copula D, as defined in (13). We will concentrate our attention on the following "dependence" properties for D.

(A) - "Independence"

$$D(x, y) = xy. \tag{15}$$

(B) - "Positive Quadrant Dependence (PQD)"

$$D(x, y) \geq xy. \tag{16}$$

(C) - "Left Tail Dependence (LTD)"

$$D(x_1, y)/x_1 \geq D(x_2, y)/x_2, \text{ with } 0 \leq x_1 \leq x_2 \leq 1. \tag{17}$$

(D) - "Totally Positivity of Order 2 (TP2)"

$$D(x_2, y_2)D(x_1, y_1) \geq D(x_1, y_2)D(x_2, y_1), \tag{18}$$

with $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$.

We claim that each of the above properties (A)-(D) has a financial meaning of its own. The case (A) is of special interest, in that it represents a subcase of the others. Hence, it is natural to view (A) as a benchmark, that will be compared with the other cases.
For our purposes, it is convenient to re-arrange the order of the discussion of the "dependence" properties listed above.

3.1 (D) - "Totally Positivity of Order 2 (TP2)"

For $x_1, x_2, y_1, y_2 > 0$, condition (18), namely:

$$exp\{-h(\psi^{-1}(-log(x_1)), -log(y_1))\} \cdot exp\{-h(\psi^{-1}(-log(x_2)), -log(y_2))\} \geq$$

$$\geq exp\{-h(\psi^{-1}(-log(x_1)), -log(y_2))\} \cdot exp\{-h(\psi^{-1}(-log(x_2)), -log(y_1))\}, \tag{19}$$

can be rewritten as

$$h(\psi^{-1}(u_1), v_1) - h(\psi^{-1}(u_2), v_1) \leq h(\psi^{-1}(u_1), v_2) - h(\psi^{-1}(u_2), v_2), \tag{20}$$

where $u_2 \leq u_1$ and $v_2 \leq v_1$.
We already know, by definition, that ψ is increasing, and so is ψ^{-1}. Hence, condition (20) has a simple interpretation. Recall the notation (3) and consider two couples of risky assets:

$$Y_1^+ = [\psi^{-1}(u_1), v_1], \qquad Y_2^+ = [\psi^{-1}(u_2), v_1]$$

and

$$Y_1^- = [\psi^{-1}(u_1), v_2] \qquad Y_2^- = [\psi^{-1}(u_2), v_2].$$

The superscripts $-$ and $+$ stand for *small* and *large* value of the inverted mean, in agreement with the hypothesis $v_2 \leq v_1$.
The G-equivalent riskless bonds will be denoted by Y_{B1}^+, Y_{B2}^+ and Y_{B1}^-, Y_{B2}^-, respectively.
Condition (20) means that the gap between the inverted means of Y_{B1}^+ and Y_{B2}^+ is smaller than that of Y_{B1}^- and Y_{B2}^-. By looking at the definition of Y_1^+, Y_2^+, Y_1^- and Y_2^-, we can conclude as follows: the higher the level of inverted mean of a risky asset (i.e.: the lower the expected return of it) the less profitable -in the sense of Definition 2.2- the reduction of its risk.

To sum up, case (D) is relevant, in that it concerns with the relation between inverted means and standard deviations when risky assets are compared with the corresponding G-equivalent riskless bonds.

We notice that, in the benchmark case (A), inequality (20) holds true as an equality:

$$h(\psi^{-1}(u_1), v_1) - h(\psi^{-1}(u_2), v_1) = h(\psi^{-1}(u_1), v_2) - h(\psi^{-1}(u_2), v_2), \qquad (21)$$

with $u_2 \leq u_1$ and $v_2 \leq v_1$. Equation (21) means that, given a couple of risky assets, an identical augment of the standard deviation generates a same augment of the inverted mean for the corresponding (i.e.: G-equivalent) riskless bonds. Substantially, if the inverted mean is assumed to be fixed, case (D) is associated to an investor who behaves as follows: s/he disregards such an inverted mean and takes into account only the augment of the risk in evaluating the growth of the inverted means of the G-equivalent riskfree bonds.

3.2 (C) - "Left Tail Dependence (LTD)"

For $x_1, x_2, y > 0$, condition (17), namely:

$$\frac{exp\{-h(\psi^{-1}(-log(x_1)), -log(y))\}}{x_1} \geq \frac{exp\{-h(\psi^{-1}(-log(x_2)), -log(y))\}}{x_2}, \qquad (22)$$

means

$$h(\psi^{-1}(-log(x_2)), -log(y)) - h(\psi^{-1}(-log(x_1)), -log(y)) \geq -log(x_2) + log(x_1)$$

or, equivalently,

$$h(\psi^{-1}(u_1), v) - h(\psi^{-1}(u_2), v) \leq u_1 - u_2, \qquad (23)$$

with $0 \leq u_2 \leq u_1$ and $v \geq 0$.

To explain the financial meaning of utility functions such that condition (23) holds, we firstly note that, by definition of functions h and ψ, we have $h(\psi^{-1}(u), 0) = u$, for each u. Therefore, (23) is a weaker condition than (21), namely we require that (21) holds true only when $v_2 = 0$.

However, this case is interesting also besides a comparison with case (D). Indeed, take the function ϕ defined in (9) and assume that $\phi(v) = \alpha$, for $\alpha \in \mathbb{R}$, i.e.: $v = \phi^{-1}(\alpha)$. The term α represents the utility level of the indifference curve containing $[0, v]$, i.e.: $[0, v] \in l_G^{(\alpha)}$.

We can state that $\phi(h(\psi^{-1}(u), 0)) = \phi(u) = \alpha$ is equivalent to $u = \phi^{-1}(\alpha)$. Hence, (23) can be rewritten as follows:

$$h(\psi^{-1}(u_1), v) - h(\psi^{-1}(u_2), v) \leq \phi^{-1}(\alpha_1) - \phi^{-1}(\alpha_2), \tag{24}$$

where $\alpha_1 \leq \alpha_2$ are such that $Y_{B1} = [0, u_1] \in l_G^{(\alpha_1)}$ and $Y_{B2} = [0, u_2] \in l_G^{(\alpha_2)}$.

Now observe that, by definition, the risky assets $Y_1 = [\psi^{-1}(u_1), 0]$ and $Y_2 = [\psi^{-1}(u_2), 0]$ are G-equivalent to Y_{B1} and Y_{B2}, respectively. Therefore, inequality (24) can be interpreted as follows: given two risky assets with the same inverted mean v, namely $Y_1^v = [\psi^{-1}(u_1), v]$ and $Y_2^v = [\psi^{-1}(u_2), v]$, the difference between the inverted means of the corresponding G-equivalent riskless bonds can be bounded from above by means of the utility levels associated to Y_1 and Y_2. We can say that case (C) states a specific relation between an objective parameter -the difference of the inverted means of two riskless bonds- and a subjective quantity -the utility levels of two risky assets which can be derived by the riskless bonds through the utility function G. Such a relation becomes of particular interest when referring to the analysis of case (A). Indeed, in this case, (24) is replaced by the stronger condition:

$$h(\psi^{-1}(u_1), v) - h(\psi^{-1}(u_2), v) = \phi^{-1}(\alpha_1) - \phi^{-1}(\alpha_2), \tag{25}$$

with $0 \leq u_2 \leq u_1$, $v \geq 0$ and $\alpha_1 \leq \alpha_2$. Therefore, case (A) states the equivalence between an improvement in the inverted means of the riskless bonds and the growth of the utility levels of related risky assets with 0 inverted means. It is worth noting that the right-hand side of equality (25) disregards the level of inverted mean v related to the risky assets Y_1^v and Y_2^v. This is an important outcome of our analysis: indeed, equality (25) allows us to treat in an identical manner variations of inverted means of riskless bonds and changes of investor's preferences. In other words, under (25), we can derive one of these quantities from the other. It is worth noting that, in this respect, the utility levels are *fully informative*, in that they provide an exact information on the relationships between the expected returns of the riskless bonds in the market.

3.3 (A) - "Independence"

The analysis of financial meaning of (A) has been already performed above, in discussing cases (C) and (D). However, some further explanations are needed. By recalling (13), we immediately see that (15) is equivalent to

$$h(\psi^{-1}(-log(x)), -log(y)) = -log(x) - log(y).$$

Therefore, (15) is verified if and only if

$$G\left(\psi^{-1}(u), v\right) = G(0, u+v), \qquad \forall\, u, v \in [0, +\infty). \tag{26}$$

By relation (26), and by definition of $\psi^{-1}(u)$, the economic meaning of condition (15) can be easily described as follows: given a risky asset $Y_R = [u_R, v_R]$ and a riskless bond $Y_B = [0, v_B]$, Y_R and Y_B are G-equivalent if and only if

$$v_B - v_R = h(u_R, 0). \tag{27}$$

The quantity $v_B - v_R$ is of special interest, in that it represents the reduction of inverted mean associated to an augment of the risk from 0 to u_R. In other words, $v_B - v_R$ is the *risk premium* to be corresponded to I in order to obtain G-equivalence between the riskless bond Y_B and the risky asset Y_R.
Thus, formula (27) provides an explicit shape of such a risk premium. The term $h(u_R, 0)$ is the inverted mean of a riskless bond G-equivalent to a hypothetical risky asset with an infinite expected return and risk equals to u_R. By recalling the meaning of $h(u_R, 0)$ (see formula (12) and the related discussion), we can say that the risk premium $v_B - v_R$ can be written by means of the absolute perception by the investor of the risk u_R.

3.4 (B) - "Positive Quadrant Dependence (PQD)"

Analogously to the previous case, a simple computation shows that (16) is equivalent to

$$G\left(\psi^{-1}(u), v\right) \leq G(0, u+v), \qquad \forall\, u, v \in [0, +\infty). \tag{28}$$

In order to interpret the economic meaning of inequality (28) consider a risky asset $Y_R = [u_R, v_R]$ and a riskless bond $Y_B = [0, v_B]$. By using the same arguments as in case (A) we can say that, under condition (28), Y_R and Y_B are equivalent only if

$$v_B - v_R \geq h(u_R, 0). \tag{29}$$

The quantity $v_B - v_R$ can be considered as the risk premium that allows Y_R and Y_B to be equivalent and (29) then states that it is greater than $h(u_R, 0)$. By comparing previous formula with (27) we argue that, given the risky asset Y_R, the equivalent riskless bond Y_B of the benchmark case (A) is never less profitable -in the sense of Definition 2.2- than that of the case (B).

4 Economic aspects and final remarks

This section is devoted to discussing a few economic aspects of the arguments presented above. The concept of mean-variance semi-copula allows us to establish an explicit relationship between the theoretical framework of semi-copulas and the setting of mean-variance theory.

In particular, starting from Definition 2.5, we can write explicitly the utility function G in terms of the semi-copula D. By simply rearranging the terms in the first line of (13) we in fact obtain:

$$G(u, v) = \phi\left(-log\left(D\left(e^{-\psi(u)}, e^{-v}\right)\right)\right). \tag{30}$$

On this basis, our paper offers a new perspective and shows a bridge between the theory of Markowitz and "dependence" properties of semi-copulas. More precisely, we discussed some specific economic behaviors of the investors that are suggested by the properties (A)-(D).

We note that none of the "dependence" properties discussed in the previous Section is, strictly speaking, a risk-aversion property of the related utility functions. Indeed, our approach only involves comparisons between couples of assets through the inverted mean. In the plane Υ, furthermore, investors' attitudes toward risk are not necessarily described by concavity/convexity of the indifference curves. Nevertheless, property (D) provides an inspiring suggestion on risk-aversion. More precisely, condition (D) states that the lower the level of the mean of a risky asset the less profitable -in the sense of Definition 2.2- the reduction of the risk of it. In this respect, we also address the reader to the economic discussion following ($G4$).

Let us finally come to a remark concerning feasibility of risky assets.

Feasibility of the risky assets is attained by restricting Υ to the admissible region $\Gamma \subset \Upsilon$. This restriction is necessary in order to develop a financially correct discussion. The admissible region is a standard concept in the classical mean-standard deviation framework, and its definition moves from the identification of an increasing curve, that is the Pareto-optimal efficient frontier. We denote by $\mathcal{E}$ the efficient frontier in our setting. According to Markowitz model, $\mathcal{E}$ is a decreasing curve contained in Υ and delimiting Γ from below.

Denote by $h_{|\Gamma}$ and $D_{|\Gamma}$ the restrictions to Γ of function h and of function D, defined in (7) and (13), respectively. Even if $D_{|\Gamma}$ is no longer a semi-copula, all the financial interpretations of the properties (A)-(D) hold also in this restricted case.

Such a restriction to Γ reflects into a corresponding reduction of the codomain of the function $h_{|\Gamma}$. This reduction is determined by the *individual optimum*. In this respect, we can argue as follows.

In the classical mean-standard deviation model, individuals' optima are contained in the efficient frontier. Each individual constructs the family of the indifference curves describing her/his preferences, and selects the optimum by searching for the tangent point between the indifference curves and the efficient frontier. The same applies in our framework. For a given individual, with preferences described by the utility function G, her/his optimum can be found as the tangent point $Y_G^\star = [u_G^\star, v_G^\star]$ between the indifference curves in (5) and the efficient frontier $\mathcal{E}$. Therefore, the function $h_{|\Gamma}$ becomes:

$$h_{|\Gamma} : \Gamma \to [h_G^\star, +\infty),$$

where $h_G^\star = h(u_G^\star, v_G^\star)$. This constitutes the restriction of the domain of function h which is needed to complete the translation of Markowitz' theory in our setting.

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