# Points in the Plane, Lines in Space

J.M. Selig

**Abstract.** The 1911 Grünwald - Blaschke mapping is reviewed from the point of view of a particular Clifford algebra. This is a mapping between the group of proper Euclidean displacements of the plane and an open set in 3-dimensional real projective space. The image of the set of group elements which displace an arbitrary point to another fixed point is a line in the projective space. In this way, a correspondence is established between point-pairs in the plane and lines in 3-dimensional projective space. The space of lines in 3 dimensions is an object of classical study usually called the Klein quadric. The action of the group of planar rigid-body displacements on the Klein quadric is different from the usually considered action of the spatial group. The quadratic invariants with respect to this representation are found and interpretations in terms of point-pairs are given. Some subspaces of lines, including line complexes and congruences, are investigated and their interpretation as sets of point-pairs in the plane are given.

**Mathematics Subject Classification (2010).** Primary 53A20; Secondary 51A10, 53A25.

In some recent work in Computational geometry, the Erdős distinct distances problem was considered. This asks: given $n$ distinct points in the plane, what is the minimum number of distinct distances we can have between the pairs of points, see [4, 11]. A key observation there was to identify pairs of points in the plane with lines in 3-dimensional space. More precisely, the group of planar rigid-body displacements can be identified with a 3-dimensional space. Then the set of group elements that transform a given point in the plane to another given point form a line in the 3-space. However, the mapping used was rather crude, a more sophisticated version was given by Tao [14] a little later. One aim of this work is to show that the 1911 mapping due to Grünwald and, independently the same year, by Blaschke see [1, chapter XI], is much neater and also leads to interesting geometry. It should be noted that this map was used in [8] to look at closely related problems.

The other aim here is to explore some of this geometry. The paper by Grünwald [3], in fact explicitly looks at the connection between point-pairs in the plane and lines in space and the mapping arises in this context. In this work, a different approach is taken. A Clifford algebra naturally associated to the group of planar rigid-body displacements is introduced. Elements of this algebra can be used to represent elements of the group, points in the plane, and also lines in the plane. The Grünwald - Blaschke mapping is then easily expressed in terms of the coefficients of the elements of the Clifford algebra. The map takes group elements to points in a 3-dimensional projective space away from a particular line. It is then shown that the set of group elements that transform a given point in the plane to another fixed point, form a line in the 3-dimensional projective space. This gives a correspondence between point-pairs in the plane and lines in space.

The action of the group of planar rigid-body displacements lifts to an action on the lines. Quadratic invariants under this action are computed and interpreted in terms of point-pairs.

The rest of the paper is then devoted to looking at the correspondences between sets of lines in space and sets of point-pairs in the plane. In particular, line complexes and congruences are investigated. Several of these results appear in [3] but without explicit proof, as was the style at the time.

## 1. Clifford Algebra

The appropriate Clifford algebra for 2-dimensional Euclidean geometry is $Cl(0,2,1)$. This algebra has two generators which square to $-1$: $e_1^2 = e_2^2 = -1$ and one which squares to zero $e^2 = 0$. These generators also anti-commute. The Clifford conjugation operation acts on any product of generators by reversing their order and making each one negative,

$$(e_1e_2e)^- = (-e)(-e_2)(-e_1) = e_1e_2e.$$

This is extended to the rest of the algebra by linearity.

In this algebra, a rigid displacement, or rather the double cover of a rigid displacement, is represented by an element with even degree. Rotations about the origin have the form,

$$\tilde{r} = \cos\frac{\theta}{2} + e_1e_2\sin\frac{\theta}{2},$$

where $\theta$ is the angle of rotation. A translation with translation vector $\vec{t} = (t_x,\, t_y)^T$, is given by the element,

$$\tilde{t} = 1 + \frac{t_x}{2}e_1e + \frac{t_y}{2}e_2e.$$

Multiplication in the group is modelled by the Clifford product. A general proper rigid-body displacement is a rotation followed by a translation and is thus given by,

$$\tilde{g} = \tilde{t}\tilde{r} = \cos\frac{\theta}{2} + e_1e_2\sin\frac{\theta}{2} + \frac{1}{2}(t_x\cos\frac{\theta}{2} + t_y\sin\frac{\theta}{2})e_1e + \frac{1}{2}(t_y\cos\frac{\theta}{2} - t_x\sin\frac{\theta}{2})e_2e.$$

The inverse of a group element is given by its Clifford conjugate,

$$\tilde{g}^{-1} = \tilde{g}^{-} = \cos\frac{\theta}{2} - e_1e_2\sin\frac{\theta}{2} - \frac{1}{2}(t_x\cos\frac{\theta}{2} + t_y\sin\frac{\theta}{2})e_1e - \frac{1}{2}(t_y\cos\frac{\theta}{2} - t_x\sin\frac{\theta}{2})e_2e.$$

Not every even grade element of the Clifford algebra represents a rigid displacement. Only those elements that satisfy the equation $\tilde{g}^{-1}\tilde{g} = 1$ are elements of the group.

A point in the plane $p = (x, y)$, can be represented by an algebra element of the form, $\tilde{p} = 1 + xe_1e + ye_2e$. Now, the action of the group element $\tilde{g}$ on the point is then given by,

$$1 + x'e_1e + y'e_2e = \tilde{g}(1 + xe_1e + ye_2e)\tilde{g}^{\dagger},$$

where

$$\tilde{g}^{\dagger} = \cos\frac{\theta}{2} - e_1e_2\sin\frac{\theta}{2} + \frac{1}{2}(t_x\cos\frac{\theta}{2} + t_y\sin\frac{\theta}{2})e_1e + \frac{1}{2}(t_y\cos\frac{\theta}{2} - t_x\sin\frac{\theta}{2})e_2e.$$

This is slightly different from the Clifford conjugate. Notice that $\tilde{g}$ and $-\tilde{g}$ will give the same result for any point in the plane. So both $\tilde{g}$ and $-\tilde{g}$ represent the same rigid displacement. This was referred to earlier, the group of unit dual quaternions double covers the proper rigid-body displacements.

The Clifford algebra also contains elements which can represent lines in the plane. These are elements of the form,

$$\pi = -de_1e_2e - n_ye_1 + n_xe_2.$$

Here $\vec{n} = (n_x, n_y)^T$ is the normal unit vector to the line and $d$ is the perpendicular oriented distance from the origin to the line. The action of the rigid displacements on these lines is the same as the action on points,

$$(-d'e_1e_2e - n'_ye_1 + n'_xe_2) = \tilde{g}(-de_1e_2e - n_ye_1 + n_xe_2)\tilde{g}^{\dagger}. \tag{1.1}$$

We can pass from this double cover group to $SE(2)$; the group of rigid displacements in the plane, by thinking of the parameters as homogeneous coordinates in an $\mathbb{RP}^3$. Taking a ray through the origin in $\mathbb{R}^4$ will identify $\tilde{g}$ and $-\tilde{g}$. So, a general rigid body displacement will be represented by elements of the form,

$$g = a_0 + a_3e_1e_2 + a_1e_1e + a_2e_2e$$

where $(a_0 : a_1 : a_2 : a_3) \in \mathbb{RP}^3$, are homogeneous coordinates. Now however, the condition, $gg^- = 1$ has no meaning. The elements of $SE(2)$ will lie in the open set where $gg^- \neq 0$. That is, the points in $\mathbb{RP}^3$ which satisfy $gg^- = 0$, do not correspond to rigid displacements. They may be considered as ideal elements. Expanding the equation gives,

$$gg^- = a_0^2 + a_3^2 = 0.$$

This has real solutions given by $a_0 = a_3 = 0$, that is a line in $\mathbb{RP}^3$. This line will be denoted $\ell_\infty$ henceforth. Thus we can identify the group manifold of $SE(2)$ with the real projective space minus a single line, $\mathbb{RP}^3 \setminus \ell_\infty$.

Notice that the ground field we are working in is crucially important here. Over the complex numbers the solution to the above equation would be a pair of complex 2-planes, meeting in $\ell_\infty$.

Now, the key observation in this work is given by the following theorem.

**Theorem 1.1.** *The set of rigid displacements which move a point $p$ to a point $q$ comprise a line in $\mathbb{RP}^3$.*

*Proof.* The line can be expressed by the linear equation in the Clifford algebra,

$$gp - q(g^\dagger)^- = 0.$$

Expanding this and comparing coefficient of the Clifford generators give the two linear equations,

$$\begin{aligned} 2a_1 - a_0(q_x - p_x) - a_3(q_y + p_y) &= 0, \\ 2a_2 - a_0(q_y - p_y) + a_3(q_x + p_x) &= 0. \end{aligned} \tag{1.2}$$

□

*Remark* 1.2. Another way to see this is as follows. Notice here that the rotations about the origin are given by the line of elements satisfying $a_1 = a_2 = 0$. Now any rigid displacement taking the point $p = 1 + p_x e_1 e + p_y e_2 e$ to a point $q = 1 + q_x e_1 e + q_y e_2 e$ can be decomposed into a translation $1 - (1/2)p_x e_1 e - (1/2)p_y e_2 e$ which translates $p$ to the origin, followed by a rotation about the origin $\cos(\theta/2) + \sin(\theta/2)e_1 e_2$ and then finally a translation which moves the origin to $q$, $1 + (1/2)q_x e_1 e + (1/2)q_y e_2 e$,

$$\begin{aligned} (1 + \frac{1}{2}q_x e_1 e + \frac{1}{2}q_y e_2 e)(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1 e_2)(1 - \frac{1}{2}p_x e_1 e - \frac{1}{2}p_y e_2 e) = \\ \cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1 e_2 + \frac{1}{2}\big((q_x - p_x)\cos\frac{\theta}{2} + (q_y + p_y)\sin\frac{\theta}{2}\big)e_1 e + \\ \frac{1}{2}\big((q_y - p_y)\cos\frac{\theta}{2} - (q_x + p_x)\sin\frac{\theta}{2}\big)e_2 e. \end{aligned} \tag{1.3}$$

It is simple to verify that the coefficients of the Clifford generators satisfy the linear equations (1.2) found above.

It is well known that almost all rigid-body displacements in the plane are rotations about a fixed centre in the plane. The exceptions being pure translations. In the Clifford algebra a rotation about a point $c = (c_x, c_y)$ can be found from a conjugation in the group. We can translate $c$ to the origin, rotate about the origin and then translate the origin back to $c$. In the Clifford algebra this can be written as $\tilde{c}\tilde{r}\tilde{c}^-$ where $\tilde{r}$ is a rotation about the origin, as above, and $\tilde{c} = 1 + (1/2)c_x e_1 e + (1/2)c_y e_2 e$. Performing the computation gives,

$$\tilde{c}\tilde{r}\tilde{c}^- = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1 e_2 + c_y \sin\frac{\theta}{2}e_1 e - c_x \sin\frac{\theta}{2}e_2 e. \tag{1.4}$$

Comparing equations (1.3) and (1.4) we can see that,

**Theorem 1.3.** *The rotation centres of the set of group elements that displace a point $p$ to $q$ lie along the perpendicular bisector line between $p$ and $q$.*

*Proof.* Comparing the coefficients of $e_1 e$ and $e_2 e$ in the two equations gives,

$$c_y \sin\frac{\theta}{2} = \frac{1}{2}\left((q_x - p_x)\cos\frac{\theta}{2} + (q_y + p_y)\sin\frac{\theta}{2}\right),$$
$$c_x \sin\frac{\theta}{2} = \frac{1}{2}\left((q_x + p_x)\sin\frac{\theta}{2} - (q_y - p_y)\cos\frac{\theta}{2}\right).$$

Multiplying the first equation by $(q_y - p_y)$, the second by $(q_x - p_x)$ and adding the equations together gives,

$$(q_x - p_x)c_x + (q_y - p_y)c_y = \frac{1}{2}(q_x^2 + q_y^2) - \frac{1}{2}(p_x^2 + p_y^2),$$

after cancelling the $\sin(\theta/2)$. This gives the equation of the line in $\mathbb{RP}^2$, on which the rotation centres lie. The normal vector to this line is given by,

$$\vec{n} = \begin{pmatrix} (q_x - p_x) \\ (q_y - p_y) \end{pmatrix}.$$

Although this vector is not unit length, we see that it is parallel to the line from $p$ to $q$ and hence normal to the perpendicular bisector between $p$ and $q$. When $\theta = \pi$ radians we have the centre of rotation located at,

$$\begin{pmatrix} c_x \\ c_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} q_x + p_x \\ q_y + p_y \end{pmatrix},$$

That is, midway between $p$ and $q$. ☐

*Remark* 1.4. This result is, of course, also easy to see by more geometrical methods.

Next we look at sets of group elements that form 2-planes in $\mathbb{RP}^3$.

**Theorem 1.5.** *A set of planar rigid displacements whose rotation centres lie on a line in the plane correspond to group elements lying on a 2-plane in $\mathbb{RP}^3$.*

*Proof.* Suppose the rotation centres of the displacements lie on a line given by,

$$Ac_x + Bc_y + C = 0,$$

where $A$, $B$ and $C$ are constants. From equation (1.4) it is clear that the group elements will satisfy the linear equation,

$$-Aa_2 + Ba_1 + Ca_3 = 0.$$

In $\mathbb{RP}^3$ a 2-plane consists of the set of points satisfying a linear equation. ☐

*Remark* 1.6. In the group there are other 2-planes consisting of other sets of displacements. For example, the set of pure translations comprise the solutions to the linear equation $a_3 = 0$ and the set of rotations with rotation angle $\pi$ satisfy $a_0 = 0$. Slightly more generally, rotations with fixed rotation angle $\psi$ will lie on the 2-plane determined by $a_3 \cos\psi - a_0 \sin\psi = 0$.

The most general 2-plane of group elements is, however, given by the following.

**Theorem 1.7.** *A general 2-plane of displacements in $\mathbb{RP}^3$ consists of rotations about each point of the plane by an angle that depends on the position of the point. The 2-plane will also determine a set of parallel lines in the plane such that rotations about points lying on the same line have the same rotation angle.*

*Proof.* A general 2-plane is given by a linear equation of the form,

$$A_0 a_0 + A_1 a_1 + A_2 a_2 + A_3 a_3 = 0, \tag{1.5}$$

where the $A_i$s are fixed coefficients. Using equation (1.4) to substitute for the coordinates of $\mathbb{RP}^3$ gives,

$$A_0 \cos\frac{\theta}{2} + A_3 \sin\frac{\theta}{2} + A_1 c_y \sin\frac{\theta}{2} - A_2 c_x \sin\frac{\theta}{2} = 0.$$

Rearranging this gives the linear equation,

$$-A_2 c_x + A_1 c_y + \left(A_3 + A_0 \cot\frac{\theta}{2}\right) = 0.$$

Clearly, for a fixed value of the rotation angle the possible rotation centres $(c_x,\, c_y)$ will lie on a line with normal vector $\vec{n} = \begin{pmatrix} -A_2 \\ A_1 \end{pmatrix}$. On the other hand. Given a point in the plane $(c_x,\, c_y)$, we can use the equation to find the rotation angle for the point,

$$\theta = 2\arctan\left(\frac{A_0}{A_2 c_x - A_1 c_y - A_3}\right).$$

□

*Remark* 1.8. In the set of parallel lines determined by the 2-plane there will be a line corresponding to a rotation angle of $\pi$. We can choose coordinates so that the origin lies on this line and the $x$-axis of our coordinates are parallel to the lines. With this choice of coordinates the coefficients will satisfy $A_2 = A_3 = 0$. In this way the $x$-axis becomes the line corresponding to rotations by $\pi$ and the parallel line a distance $h$ above the $\pi$-line corresponds to rotations by,

$$\theta = -2\arctan\left(\frac{A_0}{A_1 h}\right).$$

## 2. The Grünwald - Blaschke Mapping

The Grünwald - Blaschke mapping is implicit in the Clifford algebra representation of $SE(2)$ discussed in the previous section. In this section it will be made explicit. It is usual to represent proper rigid-body displacements in the plane by $3 \times 3$ matrices of the form,

$$G = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

This is often referred to as the homogeneous representation of the group. The action of such a displacement on a point $p = (p_x,\, p_y)$ in the plane can then be conveniently represented by the matrix-vector product,

$$\begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_x\cos\theta - p_y\sin\theta + t_x \\ p_x\sin\theta + p_y\cos\theta + t_y \\ 1 \end{pmatrix} = \begin{pmatrix} p'_x \\ p'_y \\ 1 \end{pmatrix}, \tag{2.2}$$

where $p' = (p'_x,\, p'_y)$ is the transformed position of the point. Such a displacement can be thought of as a rotation about the origin by angle $\theta$ followed by a translation given by the vector $\vec{t} = (t_x,\, t_y)^T$.

A general point in the projective space $\mathbb{RP}^3$ can be written using homogeneous coordinates as $(a_0 : a_1 : a_2 : a_3)$, where not all $a_0,\, a_1,\, a_2,\, a_3$ are zero.

The birational map from $\mathbb{RP}^3$ to $SE(2)$ is given explicitly as,

$$(a_0 : a_1 : a_2 : a_3) \longmapsto \frac{1}{a_0^2 + a_3^2} \begin{pmatrix} a_0^2 - a_3^2 & -2a_0a_3 & 2(a_0a_1 + a_3a_2) \\ 2a_0a_3 & a_0^2 - a_3^2 & 2(a_3a_1 - a_0a_2) \\ 0 & 0 & a_0^2 + a_3^2 \end{pmatrix}. \tag{2.3}$$

Clearly, multiplying the homogeneous coordinates of $\mathbb{RP}^3$ by a non-zero constant will produce the same rigid-body displacement. The exceptional set of this mapping consists of the line in $\mathbb{RP}^3$ given by $a_0 = a_3 = 0$, that is $\ell_\infty$. Away from $\ell_\infty$ the map is clearly differentiable, 1-to-1 and onto.

To write the inverse of this map it is best to think of the $3 \times 3$ matrix as,

$$\begin{pmatrix} C & -S & \tau_x \\ S & C & \tau_y \\ 0 & 0 & \Delta \end{pmatrix}$$

where the entries $C, S, \tau_x, \tau_y$ and $\Delta$ are homogeneous coordinates in a 4-dimensional projective space, $\mathbb{RP}^4$. Not every point in $\mathbb{RP}^4$ corresponds to a group elements though. It is not difficult to see that the group elements must satisfy the homogeneous equation,

$$C^2 + S^2 - \Delta^2 = 0.$$

This is a singular quadric in $\mathbb{RP}^4$, where the singularity is the line $C = S = \Delta = 0$. The matrices corresponding to points on this line are clearly not rigid-body displacements. So, in this representation the rigid-body displacements correspond to the points on the quadric minus its singular line.

**Definition 2.1.** The Grünwald - Blaschke map is the inverse to the mapping given in (2.3),

$$\begin{aligned} a_0 &= 2(C+\Delta)\Delta, \\ a_3 &= 2S\Delta, \\ a_1 &= \big((C+\Delta)\tau_x + S\tau_y\big), \\ a_2 &= \big(S\tau_x - (C+\Delta)\tau_y\big). \end{aligned} \tag{2.4}$$

As defined above, this is a quadratic map from $\mathbb{RP}^4$ to $\mathbb{RP}^3$. The image of the hyperplane $\Delta = 0$ is clearly $\ell_\infty$.

Substituting the map in (2.3) into the above gives the identity map up to an overall factor of $4a_0(a_0^2 + a_3^2)$. Since the coordinates are homogeneous, this common factor can be ignored.

The exceptional set of this map, the points where the map is undefined, consist of the 2-plane $C + \Delta = S = 0$ and the line $\tau_x = \tau_y = \Delta = 0$. The 2-plane contains all rotations by $\pi$ radians and the singular line of the quadric. The line, $\tau_x = \tau_y = \Delta = 0$, doesn't meet the quadric of rigid-body displacements — unless we change the ground field to $\mathbb{C}$. See [1, Ch. 11] and also [7] for more details.

## 3. Lines in $\mathbb{RP}^3$ - The Klein Quadric

Theorem 1.1 showed a connection between pairs of points in the plane and lines in space. The study of lines in 3-dimensions is a subject with a long and illustrious history. However, line geometry in 3-D is usually concerned with the geometry of 3-dimensional space and hence the action of the group $SE(3)$, of rigid-body displacements in spaces is often important either explicitly or implicitly. Here we are concerned with the geometry relative to the group of planar motions $SE(2)$. This does not seem to have been studied to any great extent. Hence, the action of $SE(2)$ on lines in $\mathbb{RP}^3$ will be considered in a following section. First, the geometry of the space of lines in space will be revisited in order to set-up notation and make the connection with pairs of points in the plane explicit.

Lines in $\mathbb{RP}^3$ are usually written in terms of Plücker coordinates. Consider a pair of points in the projective space $\mathbb{RP}^3$, $\bar{a} = (a_0 : a_1 : a_2 : a_3)$ and $\bar{b} = (b_0 : b_1 : b_2 : b_3)$. The Plücker coordinates of the line $\ell$ joining these two points are given by,

$$P_{ij} = \det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} = a_i b_j - a_j b_i,$$

where we assume $0 \le i < j \le 3$. Notice that if a different pair of points on the line are taken then the Plücker coordinates are unchanged except that they may be all multiplied by a non-zero constant. Hence, the six coordinates can be taken as homogeneous coordinates in $\mathbb{RP}^5$. The Plücker coordinates can be collected into a vector in the following order,

$$\ell = \begin{pmatrix} P_{13} \\ P_{23} \\ P_{12} \\ P_{20} \\ P_{01} \\ P_{03} \end{pmatrix}. \tag{3.1}$$

This order is, of course, arbitrary and the above choice is not the conventional one used when studying the geometry of 3-dimensional space. The reason for this order will become apparent in section 5 below. Also, note that $P_{20} = -P_{02}$ has been used, this avoids a minus sign later.

As a first example, consider the line $\ell_\infty$. We can think of this as the line passing through the two points $(0 : 1 : 0 : 0)$ and $(0 : 0 : 1 : 0)$ in $\mathbb{RP}^3$. Hence the Plücker coordinates of this line are given by,

$$\ell_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The main example of interest here is the line of group elements transforming point $p = (p_x, p_y)$ to a point $q = (q_x, q_y)$. A parameterisation of this line was given in (1.3) above. Using $c$ and $s$ as homogeneous parameters the line can be parameterised as,

$$(a_0 : a_1 : a_2 : a_3) = \\ (2c : c(q_x - p_x) + s(q_y + p_y) : c(q_y - p_y) - s(q_x + p_x) : 2s) \tag{3.2}$$

Two points on the line can be found by setting $c = 1$, $s = 0$ and $c = 0$, $s = 1$ for example. These points are,

$$\bar{g}_a = \big(2 : (q_x - p_x) : (q_y - p_y) : 0\big)$$

and

$$\bar{g}_b = \big(0 : (q_y + p_y) : -(q_x + p_x) : 2\big),$$

respectively. The Plücker coordinates of this line are then,

$$\ell_{pq} = \begin{pmatrix} 2(q_x - p_x) \\ 2(q_y - p_y) \\ (p_x^2 + p_y^2) - (q_x^2 + q_y^2) \\ 2(q_x + p_x) \\ 2(q_y + p_y) \\ 4 \end{pmatrix}. \tag{3.3}$$

Notice that this gives a map from directed pairs of points in the plane to lines in $\mathbb{RP}^3$. The pairs of points are directed since reversing the order of the points gives a different line,

$$\ell_{qp} = \begin{pmatrix} 2(p_x - q_x) \\ 2(p_y - q_y) \\ (q_x^2 + q_y^2) - (p_x^2 + p_y^2) \\ 2(p_x + q_x) \\ 2(p_y + q_y) \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \ell_{pq}.$$

That is, reversing the order of the points changes the sign of the Plücker coordinates, $P_{13}$, $P_{23}$ and $P_{12}$.

From this it is clear that the line of group elements which move a point $p$ to itself, that is the pure rotations about $p$, correspond to the line,

$$\ell_{pp} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p_x \\ p_y \\ 1 \end{pmatrix}.$$

Recall that these are homogeneous coordinates so a non-zero overall factor can be cancelled. Notice that a line representing rotations about a fixed point will contain the identity in the group, that is the point $(1 : 0 : 0 : 0)$. Moreover, these lines are easily seen to be subgroups of $SE(2)$.

Not all points in $\mathbb{RP}^5$ correspond to lines in $\mathbb{RP}^3$ since their coordinates must satisfy the condition,

$$P_{01}P_{23} + P_{20}P_{13} + P_{03}P_{12} = 0. \tag{3.4}$$

Using $P_{02}$ rather than $P_{20}$ would introduce a minus sign into the above equation. This condition can be found by expanding the following $4 \times 4$ determinant identity,

$$\det \begin{pmatrix} a_0 & b_0 & a_0 & b_0 \\ a_1 & b_1 & a_1 & b_1 \\ a_2 & b_2 & a_2 & b_2 \\ a_3 & b_3 & a_3 & b_3 \end{pmatrix} = 0.$$

This equation defines a 4-dimensional variety of degree 2 in $\mathbb{RP}^5$, it is usually known as the Klein quadric or sometimes Plücker's quadric. The lines of $\mathbb{RP}^3$ are in 1-to-1 correspondence with the points of this quadric.

*Remark* 3.1. It was shown here that the set of group elements taking a point $p$ to a point $q$ comprise a line in the $\mathbb{RP}^3$ representing $SE(2)$. How does this line of displacements move other points in the plane? The line of displacements $\ell_{pq}$ that move a point $p$ in the plane to a point $q$ will also move any circle centred on $p$ to a circle centred on $q$ with the same radius. Another way of expressing this it to say that any point a distance $r$ from $p$ will be moved to a point that is a distance $r$ from $q$. This can be easily seen from the description of $\ell_{pq}$ given above.

## 4. More on Lines in $\mathbb{RP}^3$

Not all lines in $\mathbb{RP}^3$ correspond to point-pairs in the plane. To see this consider how one might recover the points given a line. Suppose the line is given by $\ell$ as in equation (3.1). Comparing this to the Plücker coordinates of the line of displacements taking $p$ to $q$, as given in equation (3.3), we can see that,

$$p_x = \frac{P_{20} - P_{13}}{P_{03}}, \quad p_y = \frac{P_{01} - P_{23}}{P_{03}}, \tag{4.1}$$

and the second point is given by,

$$q_x = \frac{P_{20} + P_{13}}{P_{03}}, \quad q_y = \frac{P_{01} + P_{23}}{P_{03}}. \tag{4.2}$$

This gives a unique pair of finite points as long as $P_{03} \neq 0$. So, lines with Plücker coordinate $P_{03} = 0$ do not correspond to point-pairs in the plane.

This leads to the following.

**Theorem 4.1.** *The set of rigid displacements which move a line $\varpi$ to a line $\pi$ is given by a line in $\mathbb{RP}^3$ with Plücker coordinate $P_{03} = 0$.*

*Proof.* From (1.1), a rotation that moves the $x$-axis $-e_1$, to the line,

$$\pi = -de_1e_2e - n_ye_1 + n_xe_2$$

is given by,

$$g = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}e_1e_2 - \frac{d}{2}\sin\frac{\phi}{2}e_1e + \frac{d}{2}\cos\frac{\phi}{2}e_2e,$$

where, the angle the line makes with the $x$-axis is $\phi$. So that, $n_x = -\sin\phi = -2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$ and $n_y = \cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}$. Now, given two lines in the plane,

$$\pi = -de_1e_2e - n_ye_1 + n_xe_2 \quad \text{and} \quad \varpi = -he_1e_2e - m_ye_1 + m_xe_2,$$

we can think of the group elements that take $\varpi$ to $\pi$ as consisting of a rotation $g_2^-$, moving $\varpi$ into coincidence with the $x$-axis, followed by an arbitrary translation $s_x$ in the $x$-direction and then finally a rotation $g_1$ taking the $x$-axis into coincidence with $\pi$. In the Clifford algebra this can be written as $g_1s_xg_2^-$, which can be expanded to,

$$\begin{aligned}
(c_1 + s_1e_1e_2 - \frac{ds_1}{2}e_1e + \frac{dc_1}{2}e_2e)(1+\gamma e_1e)(c_2 - s_2e_1e_2 + \frac{hs_2}{2}e_1e - \frac{hc_2}{2}e_2e)& \\
= (c_1c_2 + s_1s_2) + (s_1c_2 - c_1s_2)e_1e_2 +& \\
\big(\gamma(c_1c_2 - s_1s_2) - \frac{d-h}{2}(s_1c_2 + c_1s_2)\big)e_1e +& \\
\big(\gamma(s_1c_2 + c_1s_2) + \frac{d-h}{2}(c_1c_2 - s_1s_2)\big)e_2e,&
\end{aligned}$$

with $c_1 = \cos\frac{\phi_1}{2}$ and $s_1 = \sin\frac{\phi_1}{2}$ and similar for $c_2$ and $s_2$, and where $\gamma$ is an affine parameter. This is an affine parameterisation of a line in $\mathbb{RP}^3$. We can get a homogeneous parameterisation of the line by substituting $\gamma = \mu/\lambda$

and multiply through by $\lambda$. Then we can look at the two points on the line given by $\lambda = 1$, $\mu = 0$,

$$\bar{g}_a = \left(\cos\left(\frac{\phi_1 - \phi_2}{2}\right) : -\frac{d-h}{2}\sin\left(\frac{\phi_1 + \phi_2}{2}\right) : \right.$$
$$\left. \frac{d-h}{2}\cos\left(\frac{\phi_1 + \phi_2}{2}\right) : \sin\left(\frac{\phi_1 - \phi_2}{2}\right)\right)$$

and when $\lambda = 0$ and $\mu = 1$ we have,

$$\bar{g}_b = \left(0 : \cos\left(\frac{\phi_1 + \phi_2}{2}\right) : \sin\left(\frac{\phi_1 + \phi_2}{2}\right) : 0\right).$$

Finally, the Plücker coordinates for this line are then,

$$\ell_{\varpi\pi} = \begin{pmatrix} -\sin\phi_1 + \sin\phi_2 \\ \cos\phi_1 - \cos\phi_2 \\ h - d \\ -\sin\phi_1 - \sin\phi_2 \\ \cos\phi_1 + \cos\phi_2 \\ 0 \end{pmatrix} = \begin{pmatrix} n_x - m_x \\ n_y - m_y \\ h - d \\ n_x + m_x \\ n_y + m_y \\ 0 \end{pmatrix}.$$

Recall again that, Plücker coordinates are homogeneous coordinates, so multiplying by an overall constant has no effect. □

*Remark* 4.2. Note that any line parallel to $\varpi$ a distance say, $r$ from it will be displaced to a line parallel to $\pi$ a distance $r$ from $\pi$. So, we cannot associate a single pair of lines in the plane to such a line in $\mathbb{RP}^3$.

## 5. The action of $SE(2)$ on Lines in $\mathbb{RP}^3$

Above, in equation (2.2), we saw the action of elements of the group of rigid-body motions in the plane on points in the plane. This can be extended to an action on lines in $\mathbb{RP}^3$.

**Theorem 5.1.** *The action of $SE(2)$ on lines in $\mathbb{RP}^3$ is the direct sum of the standard homogeneous representation of the group with its dual or inverse transpose representation.*

*Proof.* Suppose both points $p$ and $q$ are subject to a displacement,

$$\begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_x\cos\theta - p_y\sin\theta + t_x \\ p_x\sin\theta + p_y\cos\theta + t_y \\ 1 \end{pmatrix},$$

and similar for $q$. Then we will have that,

$$\begin{aligned} P_{13} = 2(q_x - p_x) &\mapsto 2(q_x - p_x)\cos\theta - 2(q_y - p_y)\sin\theta \\ &= \cos\theta P_{13} - \sin\theta P_{23}, \\ P_{23} = 2(q_y - p_y) &\mapsto 2(q_x - p_x)\sin\theta + 2(q_y - p_y)\cos\theta \\ &= \sin\theta P_{13} + \cos\theta P_{23}. \end{aligned}$$

We also have that,

$$\begin{aligned} P_{20} = 2(q_x + p_x) \mapsto\, & 2(q_x + p_x)\cos\theta - 2(q_y + p_y)\sin\theta + 4t_x \\ & = \cos\theta P_{20} - \sin\theta P_{01} + t_x P_{03}, \\ P_{01} = 2(q_y + p_y) \mapsto\, & 2(q_x + p_x)\sin\theta + 2(q_y + p_y)\cos\theta + 4t_y \\ & = \sin\theta P_{20} + \cos\theta P_{01} + t_y P_{03}. \end{aligned}$$

After a little algebra, we get that,

$$\begin{aligned} P_{12} = \big((p_x^2 + p_y^2) - (q_x^2 + q_y^2)\big) \mapsto & \\ P_{12} - (t_x\cos\theta + t_y\sin\theta)P_{13} - (-t_x\sin\theta + t_y\cos\theta)P_{23} & \end{aligned}$$

This can be written in matrix form as,

$$\begin{pmatrix} P_{13} \\ P_{23} \\ P_{12} \\ P_{20} \\ P_{01} \\ P_{03} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ \rho_x & \rho_y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & t_x \\ 0 & 0 & 0 & \sin\theta & \cos\theta & t_y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_{13} \\ P_{23} \\ P_{12} \\ P_{20} \\ P_{01} \\ P_{03} \end{pmatrix},$$

where,

$$\begin{aligned} \rho_x &= -(t_x\cos\theta + t_y\sin\theta), \\ \rho_y &= -(-t_x\sin\theta + t_y\cos\theta). \end{aligned}$$

Notice that we can write,

$$(\rho_x,\, \rho_y) = -(t_x,\, t_y)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = -\vec{t}^{\,T} R.$$

The $6 \times 6$ matrix above can be written in partitioned form as,

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ \rho_x & \rho_y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & t_x \\ 0 & 0 & 0 & \sin\theta & \cos\theta & t_y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} G^{-T} & 0 \\ 0 & G \end{pmatrix},$$

where 0 represents the $3 \times 3$ zero matrix here, and $G$ is the $3 \times 3$ standard representation of $SE(2)$ as given in equation (2.1). The matrix $G^{-T}$ is the inverse, transpose of $G$, that is,

$$G^{-T} = \left(G^{-1}\right)^T = \left(G^T\right)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \rho_x & \rho_y & 1 \end{pmatrix}.$$

□

*Remark* 5.2. The six Plücker coordinates can be partitioned into two sets of 3. The group action shows that the first three represent a line in the plane while the last three correspond to a point. This is the justification for the unconventional order used for the Plücker coordinates. The coordinates,

$$\begin{pmatrix} P_{20} \\ P_{01} \\ P_{03} \end{pmatrix} = 4 \begin{pmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ 1 \end{pmatrix},$$

represent the midpoint of the pair of points in the plane and

$$\begin{pmatrix} P_{13} \\ P_{23} \\ P_{12} \end{pmatrix} = 4 \begin{pmatrix} \frac{1}{2}(q_x - p_x) \\ \frac{1}{2}(q_y - p_y) \\ \frac{1}{4}\big((p_x^2 + p_y^2) - (q_x^2 + q_y^2)\big) \end{pmatrix},$$

represents the line in the plane perpendicular to the line determined by the point-pair. The relation for the Klein quadric (3.4), shows that the perpendicular line passes through the mid-point of the point-pair. That is, the line is the perpendicular bisector of the line joining the points, see also theorem 1.3.

This gives us another interpretation of the lines in $\mathbb{RP}^3$ as pointed lines in the plane. In this interpretation the Klein quadric represents the flag manifold of pointed lines.

*Remark* 5.3. Notice that, this $6 \times 6$ representation of the group $SE(2)$ is different from the standard representation of the group of rigid-body displacements in 3D. That is, the adjoint representation of $SE(3)$ restricted to a planar subgroup $SE(2)$. Hence, we can expect that the invariants under the two representations will be different.

So next we look at the possible quadratic invariants of these lines in $\mathbb{RP}^3$. Suppose that $\ell$ is a line in $\mathbb{RP}^3$, then a quadratic invariant will have the form,

$$\ell^T Q \ell,$$

where $Q$ is a $6 \times 6$ symmetric matrix.

**Theorem 5.4.** *A general quadratic in the Plücker coordinates of a line in $\mathbb{RP}^3$, invariant under the action of $SE(2)$ is a linear combination of three basis invariants,*

$$Q = \lambda Q_0 + \mu Q_1 + \nu Q_2$$

*where $\lambda$, $\mu$ and $\nu$ are arbitrary and the basic invariants are given by the $6 \times 6$ symmetric matrices,*

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

*and*

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* The fact that a symmetric matrix $Q$ is invariant under a rigid-body displacement in the plane is expressed by the relation,

$$\begin{pmatrix} G^{-T} & 0 \\ 0 & G \end{pmatrix}^T Q \begin{pmatrix} G^{-T} & 0 \\ 0 & G \end{pmatrix} = Q.$$

That is, if $\ell$ is subject to a rigid displacement the invariant will not change. This relation can be rewritten as,

$$\begin{pmatrix} G^{-T} & 0 \\ 0 & G \end{pmatrix}^T Q - Q \begin{pmatrix} G^{-T} & 0 \\ 0 & G \end{pmatrix}^{-1} = 0,$$

that is,

$$\begin{pmatrix} G^{-1} & 0 \\ 0 & G^T \end{pmatrix} Q - Q \begin{pmatrix} G^T & 0 \\ 0 & G^{-1} \end{pmatrix} = 0.$$

We can write the $6 \times 6$ symmetric matrix $Q$, in partitiond form as,

$$Q = \begin{pmatrix} K_\alpha & K_\beta \\ K_\beta^T & K_\gamma \end{pmatrix},$$

where $K_\alpha$ and $K_\gamma$ are $3 \times 3$ symmetric matrices but $K_\beta$ is an arbitrary $3 \times 3$ matrix. Substituting this into the equation for invariants gives 3 linear equations for the $K$ sub-matrices,

$$\begin{aligned} G^{-1}K_\alpha - K_\alpha G^T &= 0, \\ K_\beta G - GK_\beta &= 0, \\ G^T K_\gamma - K_\gamma G^{-1} &= 0. \end{aligned}$$

These equations are simple to solve for arbitrary group elements $G$, and give the solution stated. □

*Remark* 5.5. The relation for the Klein quadric, equation (3.4), can be written as,

$$\ell^T Q_0 \ell = 0.$$

Suppose we have two pairs of points in the plane, $p$, $q$ and $p'$, $q'$ and suppose the corresponding lines are,

$$\ell_{pq} = \begin{pmatrix} 2(q_x - p_x) \\ 2(q_y - p_y) \\ (p_x^2 + p_y^2) - (q_x^2 + q_y^2) \\ 2(q_x + p_x) \\ 2(q_y + p_y) \\ 4 \end{pmatrix} \text{ and } \ell_{p'q'} = \begin{pmatrix} 2(q'_x - p'_x) \\ 2(q'_y - p'_y) \\ ({p'_x}^2 + {p'_y}^2) - ({q'_x}^2 + {q'_y}^2) \\ 2(q'_x + p'_x) \\ 2(q'_y + p'_y) \\ 4 \end{pmatrix}.$$

After a little algebra we get,

$$\ell_{pq}^T Q_0 \ell_{p'q'} = 4\big((p_x - p'_x)^2 + (p_y - p'_y)^2 - (q_x - q'_x)^2 - (q_y - q'_y)^2\big).$$

This implies that $\ell_{pq}^T Q_0 \ell_{p'q'} = 0$ if and only if the distance between the points $p$ and $p'$ is the same as the distance between $q$ and $q'$. On the other hand, if the distances between these points is the same then there is a rigid-body displacement that simultaneously takes $p$ to $q$ and $p'$ to $q'$. That is, the lines $\ell_{pq}$ and $\ell_{p'q'}$ share a common point. In this way we convert a question about distances between points in the plane; a question in metric geometry, into a question about lines meeting in projective 3-dimensional space; a question in projective geometry.

*Remark* 5.6. Next we look at the interpretation of the invariant $Q_1$. A simple computation shows that,

$$\ell_{pq}^T Q_1 \ell_{p'q'} = 4\big((p_x - q_x)(p'_x - q'_x) + (p_y - q_y)(p'_y - q'_y)\big).$$

This quantity will vanish if and only if the two pairs of lines in the plane determine perpendicular directions. That is, the vectors from $p$ to $q$ is orthogonal to the vector from $p'$ to $q'$.

*Remark* 5.7. Finally here, it is easy to see that $\ell^T Q_2 \ell = 0$ if and only if $\ell$ corresponds to pairs of lines in the plane rather than a pair of points. See theorem 4.1.

## 6. $\alpha$-Planes and $\beta$-Planes

The Klein quadric is a 4-dimensional quadric in $\mathbb{RP}^5$ and it has two 'rulings'. That is, there are two families of 2-planes which lie entirely in the quadric. These are usually called $\alpha$-planes and $\beta$-planes. To see how these occur let,

$$\pi = \begin{pmatrix} P_{13} \\ P_{23} \\ P_{12} \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} P_{20} \\ P_{01} \\ P_{03} \end{pmatrix}.$$

Now, consider the matrix equations,

$$(I - M)p + (I + M)\pi = 0,$$

where $M$ is a $3 \times 3$ orthogonal matrix, that is $M^T M = I$. It is not hard to see that, for a fixed matrix $M$, the 2-planes determined by these equations lie entirely in the Klein quadric. When $\det(M) = 1$ the corresponding 2-plane is called an $\alpha$-plane and when $\det(M) = -1$, it is a $\beta$-plane. The details of this can be found in [12, §6.3].

The lines in an $\alpha$-plane consist of all the lines passing through a point in $\mathbb{RP}^3$, see [10, Lemma 2.1.9]. In 3D geometry, the $\alpha$-planes correspond to "point-stars", the set of lines passing through a fixed point. Here, the lines all contain a common group element. In terms of point-pairs in the plane this corresponds to a set of point-pairs where there is a single displacement that takes the first point to the second for each pair. In [1] points with this property are referred to as homologous points.

**Theorem 6.1.** *The point-pairs in the plane corresponding to an $\alpha$-plane in the Klein quadric lie on circles centred on a point in the plane and each pair of points subtends the same angle at the centre of the circle.*

*Proof.* If the centre of the circles is $c$ and the angle subtended by the point-pairs is $\theta$, then the group element which moves the first point of each pair to the second is a rotation about $c$ by the angle $\theta$. □

Notice that the perpendicular bisectors of the point-pairs will all pass through $c$.

Next we turn to the $\beta$-planes. In general, a $\beta$-plane in the Klein quadric corresponds to a set of lines in $\mathbb{RP}^3$ lying on a 2-plane, see [10, Lemma 2.1.9] once more. We can say something about how $\beta$-planes correspond to pairs of points in the plane. First, recall from theorem 1.7 that a general 2-plane in $\mathbb{RP}^3$ corresponds to rotations about points in the plane where the angle of rotation is constant along parallel lines in the plane. From this it is clear that for a line of group elements to lie in a 2-plane in $\mathbb{RP}^3$ the midpoint of the corresponding point-pair must lie on the line with rotation angle $\pi$.

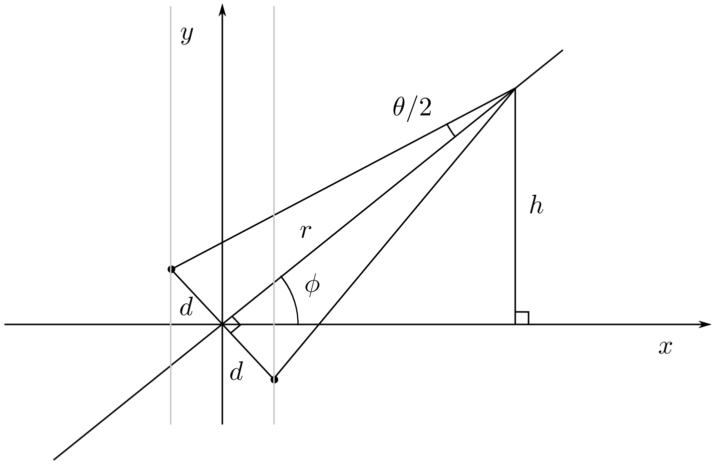


FIGURE 1. A Pair of points in the Plane.

**Theorem 6.2.** *The midpoints of point-pairs corresponding to the lines comprising a $\beta$-plane lie on a fixed line. The points of the point-pair lie on parallel lines perpendicular to the line of midpoints.*

*Proof.* Choose coordinates so that the $x$-axis coincides with the line of $\pi$ rotations determined by the 2-plane in $\mathbb{RP}^3$ given by the $\beta$-plane. Consider a pair of points whose midpoint lies at the origin of these coordinates. Remember that the line of group elements taking the first point of the pair to the second have their rotation centres on the perpendicular bisector of the line joining the points, see theorem 1.3. Suppose that this perpendicular bisector makes an angle $\phi$ with the $x$-axis and that the distance between the points is $2d$, see figure 1. From theorem 1.7 we have that,

$$\cot\frac{\theta}{2} = -\frac{A_1 h}{A_0}$$

where $A_0$ and $A_1$ are the coefficients defining the $\beta$-plane in $\mathbb{RP}^3$. The diagram, figure 1, shows that $\cot\theta/2 = r/d$ and that $h = r\sin\phi$, where $r$ is the distance between the mid-point of the point-pair and the centre of rotation. The $x$-coordinate of the lower point in the diagram is $x = d\sin\phi$ and hence, combining these results gives

$$x = d\sin\phi = \frac{dh}{r} = -\frac{A_0}{A_1}.$$

So this point lies on the line $x = -A_0/A_1$ and by symmetry the other point lies on the line $x = A_0/A_1$. $\square$

## 7. Linear Line Complexes

A complex of lines is a 3-dimensional family of lines in $\mathbb{RP}^3$. The simplest complex is a linear complex, this consists of the set of lines formed by the intersection of the Klein quadric and a 4-dimensional plane in $\mathbb{RP}^5$.

Several, sets of point-pairs correspond to linear line complexes. As a first example consider the set of point-pairs where the initial point of the pair lies on a line. If the line has normal vector $\vec{n} = (n_x, n_y)^T$ and $d$ is its perpendicular oriented distance to the origin, then the condition for the first point to lie on this line will be,

$$n_x p_x + n_y p_y - d = 0.$$

Then, using equation (4.1) to substitute for the coordinates of the point we get,

$$n_x(P_{20} - P_{13}) + n_y(P_{01} - P_{23}) - dP_{03} = 0.$$

This is the equation of a 4-plane in $\mathbb{RP}^5$.

Similarly, if the final point of the pair lies on the line we get a linear line complex determined by the 4-plane,

$$n_x(P_{20} + P_{13}) + n_y(P_{01} + P_{23}) - dP_{03} = 0.$$

If the midpoint of the pair lies on the line, the linear line complex is,

$$n_x P_{20} + n_y P_{01} - dP_{03} = 0.$$

We also get a linear complex of lines from point-pairs whose perpendicular bisectors pass through a fixed point. If the fixed point is $(x, y)$, the linear line complex is given by,

$$xP_{13} + yP_{23} + P_{12} = 0.$$

The equation for any linear line complex can be written as,

$$\ell^T Q_0 s = 0,$$

that is, the line complex is the set of all lines $\ell$ satisfying the above equation. The quantity $s$ here is a point in $\mathbb{RP}^5$, not necessarily a point representing a line in the group.

The four examples above are given respectively by,

$$s = \begin{pmatrix} n_x \\ n_y \\ -d \\ -n_x \\ -n_y \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} n_x \\ n_y \\ -d \\ n_x \\ n_y \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} n_x \\ n_y \\ -d \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \\ y \\ 1 \end{pmatrix}.$$

In the first two examples $s$ is not a line in $\mathbb{RP}^3$ but in the last two it is. A linear line complex where $s$ is itself a line is called a special linear complex. Such a complex consists of all the lines in $\mathbb{RP}^3$ meeting the line $s$.

A general linear line complex would have an arbitrary $s$, that is,

$$s = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ a_0 \\ b_0 \\ c_0 \end{pmatrix}$$

where the entries here are arbitrary but fixed. The equation of the resulting linear complex is then,

$$a_0 P_{13} + b_0 P_{23} + c_0 P_{12} + a_1 P_{20} + b_1 P_{01} + c_1 P_{03} = 0. \tag{7.1}$$

This leads to,

**Theorem 7.1.** *A general linear line complex determines a pair of points $c = (c_x, c_y)$ and $k = (k_x\, k_y)$. Fixing the initial point in a point-pair corresponding to the complex, the final point will lie on a circle centred on $k$ with radius depending on the distance of the initial point from $c$. Alternatively, if the final point of the pair is fixed, then the initial point must lie on a circle centred on $c$ with radius determined by the distance of the final point from $k$.*

*Proof.* Using equation (3.3) to substitute for the Plücker coordinates in (7.1) gives the equation,

$$c_0\big((p_x^2 + p_y^2) - (q_x^2 + q_y^2)\big) + 2\big(a_0(q_x - p_x) + a_1(q_x + p_x)\big) + \\ 2\big(b_0(q_y - p_y) + b_1(q_y + p_y)\big) + 4c_1 = 0.$$

Rearranging and completing the squares this can be written as,

$$c_0\Big(\big((p_x - c_x)^2 + (p_y - c_y)^2\big) - \big((q_x - k_x)^2 + (q_y - k_y)^2\big)\Big) = D.$$

Where,

$$c_x = \frac{(a_0 - a_1)}{c_0}, \qquad c_y = \frac{(b_0 - b_1)}{c_0},$$
$$k_x = \frac{(a_0 + a_1)}{c_0}, \qquad k_y = \frac{(b_0 + b_1)}{c_0}$$

and

$$D = -4\frac{(a_0a_1 + b_0b_1 + c_0c_1)}{c_0}.$$

This shows how the coordinates of the points $c = (c_x, c_y)$ and $k = (k_x, k_y)$ are related to the parameters of the complex. □

*Remark* 7.2. If $s$ is a line, so that the line complex is a special line complex, then $a_0a_1 + b_0b_1 + c_0c_1 = 0$ and hence $D = 0$. This implies that, in such a complex, the distance of $p$ from $c$ will be the same as the distance of $q$ from $k$.

*Remark* 7.3. Notice that the range of different linear line complexes is richer than the spatial case of lines in $\mathbb{RP}^3$. In the standard case, line complexes are classified up to the action of $SE(3)$ by the pitch of the element $s$, with pitch 0 corresponding to the special linear complex. Since the action of the planar group $SE(2)$ has different invariants, the classification of line complexes with respect to this action will be different. In particular, notice that the first two examples in this section satisfy the relation $s^TQ_2s = 0$. There is also a very special linear complex; the set of lines meeting the line $\ell_\infty$. In [3], this complex is denoted $\mathfrak{D}$. This is however, another way of expressing the hyperplane $P_{30} = 0$ of lines that don't correspond to any point-pairs, see theorem 4.1.

## 8. Quadratic Line Complexes

A quadratic line complex is a 3-dimensional set of lines in $\mathbb{RP}^3$ given by the intersection of the Klein quadric with another 4-dimensional quadric.

Clearly, from the previous section, we will get a quadratic line complex by considering the point-pairs in the plane where either the initial or final point of the pair lies on a conic curve in the plane. For example, consider the set of point-pairs in the plane where the initial point lies on a circle given by the equation,

$$x^2 + y^2 = r^2,$$

and where the radius $r$ is constant. Substituting for $x$ and $y$ using equation (4.1) gives,

$$(P_{20} - P_{13})^2 + (P_{01} - P_{23})^2 = r^2P_{03}^2.$$

This is certainly a quadratic in the Plücker coordinates and hence represents a quadratic line complex. We can also write this as $\ell^TQ\ell = 0$, were, as usual,

$\ell$ is the vector of Plücker coordinates and $Q$ is the $6 \times 6$ symmetric matrix,

$$Q = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r^2 \end{pmatrix}.$$

In fact, this is a singular quadratic complex since $\det(Q) = 0$. It is singular on the 2-plane spanned by,

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \ell_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly, we get a similar result if the final point of the pair is required to lie on a circle, in fact, only a few signs are changed. We also get very similar results if the initial or final points of the pair lie on any conic section.

As another example of a quadratic complex of lines, consider the set of all pairs of points in the plane separated by a fixed distance $2\delta$.

**Theorem 8.1.** *The lines in $\mathbb{RP}^3$ corresponding to point-pairs in the plane separated by a distance $2\delta$ comprise a singular quadratic line complex.*

*Proof.* The distance between the points $p$ and $q$ is,

$$(p_x - q_x)^2 + (p_y - q_y)^2 = 4\delta^2.$$

From equation (3.3) we have that $P_{13} = 2(q_x - p_x)$, $P_{23} = 2(q_y - p_y)$ and $P_{03} = 4$. So, substituting in the above equation gives,

$$P_{13}^2 + P_{23}^2 = \delta^2 P_{03}^2.$$

Again, this is a quadratic in the Plücker coordinates and hence a quadratic line complex. The complex is given by the $6 \times 6$ symmetric matrix,

$$Q_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta^2 \end{pmatrix}.$$

□

*Remark* 8.2. This quadratic equation can also be written in terms of the invariants found in theorem 5.4 as,

$$\ell^T Q_1 \ell - \delta^2 \ell^T Q_2 \ell = 0 = \ell^T \left(Q_1 - \delta^2 Q_2\right) \ell.$$

That is $Q_\delta = Q_1 - \delta^2 Q_2$. This determines a singular quadric in $\mathbb{RP}^5$, it is easy to see that the singular set of the quadric is the 2-plane determined by $P_{13} = P_{23} = P_{03} = 0$. Notice that, this 2-plane lies entirely within the Klein quadric. In fact this is a $\beta$-plane given by the orthogonal matrix,

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

see section 6.

This gives us a way to determine when a pair of lines represent point-pairs with the same separation.

**Theorem 8.3.** *Two lines $\ell_{pq}$ and $\ell_{p'q'}$ represent point-pairs with the same separation if,*

$$\left(\ell_{pq}^T Q_1 \ell_{pq}\right)\left(\ell_{p'q'}^T Q_2 \ell_{p'q'}\right) = \left(\ell_{p'q'}^T Q_1 \ell_{p'q'}\right)\left(\ell_{pq}^T Q_2 \ell_{pq}\right).$$

*Proof.* This relation results from eliminating $\delta^2$ between the relations,

$$\ell_{pq}^T \left(Q_1 - \delta^2 Q_2\right) \ell_{pq} = 0 \quad \text{and} \quad \ell_{p'q'}^T \left(Q_1 - \delta^2 Q_2\right) \ell_{p'q'} = 0.$$

□

*Remark* 8.4. Unlike the result in remark 5.5, this gives a direct way to compare the separation of point-pairs.

To reconcile these two views we have,

**Theorem 8.5.** *For two pairs of points in the plane, $p$, $q$ and $p'$, $q'$ the two conditions,*

$$\left(\ell_{pq}^T Q_1 \ell_{pq}\right)\left(\ell_{p'q'}^T Q_2 \ell_{p'q'}\right) - \left(\ell_{p'q'}^T Q_1 \ell_{p'q'}\right)\left(\ell_{pq}^T Q_2 \ell_{pq}\right) = 0$$

*and*

$$\left(\ell_{pp'}^T Q_0 \ell_{qq'}\right) = 0$$

*are equivalent. Both conditions are satisfied if and only if $|p - q| = |p' - q'|$.*

*Proof.* Using equation (3.3) to substitute for the coordinates of the points into the first condition gives, after a lengthy computation,

$$\left(\ell_{pq}^T Q_1 \ell_{pq}\right)\left(\ell_{p'q'}^T Q_2 \ell_{p'q'}\right) - \left(\ell_{p'q'}^T Q_1 \ell_{p'q'}\right)\left(\ell_{pq}^T Q_2 \ell_{pq}\right) = \\ 64\left((p_x - q_x)^2 + (p_y - q_y)^2 - (p'_x - q'_x)^2 - (p'_y - q'_y)^2\right)$$

When the coordinates are real, this vanishes if and only if $|p-q|^2 = |p'-q'|^2$.

A similar computation for the other condition then shows,

$$16\left(\ell_{pp'}^T Q_0 \ell_{qq'}\right) = \left(\ell_{pq}^T Q_1 \ell_{pq}\right)\left(\ell_{p'q'}^T Q_2 \ell_{p'q'}\right) - \left(\ell_{p'q'}^T Q_1 \ell_{p'q'}\right)\left(\ell_{pq}^T Q_2 \ell_{pq}\right).$$

□

*Remark* 8.6. This quadratic complex gives another realisation of the group of planar rigid displacements. Suppose we take a particular point-pair in the plane, say the points,

$$p = \begin{pmatrix} -\delta \\ 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} \delta \\ 0 \end{pmatrix}.$$

This will give a line in $\mathbb{RP}^3$ with Plücker coordinates, $\ell_{pq}^T = (\delta, 0, 0, 0, 0, 1)$. Now, acting on this line with every element of the group will produce,

$$\begin{pmatrix} G^{-T} & 0 \\ 0 & G \end{pmatrix} \ell_{pq} = \frac{1}{a_0^2 + a_3^2} \begin{pmatrix} (a_0^2 - a_3^2)\delta \\ 2a_0a_3\delta \\ -(a_0a_1 + a_3a_2)\delta \\ a_0a_1 - a_3a_2 \\ a_0a_2 + a_3a_1 \\ a_0^2 + a_3^2 \end{pmatrix}.$$

This can be thought of as a birational map from $\mathbb{RP}^3$ to the quadratic line complex in $\mathbb{RP}^5$. Explicitly, we have,

$$\begin{aligned} P_{13} &= (a_0^2 - a_3^2)\delta, & P_{20} &= a_0a_1 - a_3a_2, \\ P_{23} &= 2a_0a_3\delta, & P_{01} &= a_0a_2 + a_3a_1, \\ P_{12} &= -(a_0a_1 + a_3a_2)\delta, & P_{03} &= a_0^2 + a_3^2. \end{aligned}$$

The exceptional set of this map, that is, the set where the map is undefined, is just the ideal line $\ell_\infty$. The inverse of the map is then given by,

$$\begin{aligned} a_0 &= (\delta P_{03} + P_{13})P_{23}, & a_1 &= (\delta P_{20} - P_{12})P_{23}, \\ a_3 &= P_{23}^2, & a_2 &= -(\delta P_{03} + P_{13})(\delta P_{20} + P_{12}). \end{aligned}$$

Note that these are maps between projective spaces, so common factors can be cancelled. The exceptional set of the inverse map consists of a pair of 3-planes: $P_{23} = (\delta P_{03} + P_{13}) = 0$ and $P_{23} = (\delta P_{20} + P_{12}) = 0$.

## 9. Line Congruences

Line congruences are 2-dimensional families of lines. Often these are given by the intersection of the Klein quadric with a 3-dimensional space in $\mathbb{RP}^5$. Linear congruence are given by the intersection of the Klein quadric with 3-planes.

For example, consider the lines representing point-pairs in the plane where the first point lies on a fixed line and the second point lies on another fixed line. These conditions define a 3-plane in $\mathbb{RP}^5$ given by the equations,

$$n_x(P_{20} - P_{13}) + n_y(P_{01} - P_{23}) - dP_{03} = 0$$

and

$$n'_x(P_{20} + P_{13}) + n'_y(P_{01} + P_{23}) - d'P_{03} = 0,$$

where the normals to the lines for the first and second point are respectively,

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} \quad \text{and} \quad \vec{n}' = \begin{pmatrix} n'_x \\ n'_y \end{pmatrix},$$

and their perpendicular oriented distances from the origin are $d$ and $d'$, see section 7.

Notice that these linear equations can be written as $s_1^T Q_0 \ell = 0$ and $s_2^T Q_0 \ell = 0$ with,

$$s_1 = \begin{pmatrix} n_x \\ n_y \\ -d \\ -n_x \\ -n_y \\ 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} n'_x \\ n'_y \\ -d' \\ n'_x \\ n'_y \\ 0 \end{pmatrix}.$$

In [10, Chap. 7], linear line complexes are classified as: elliptic, hyperbolic or parabolic. If the line $s = \lambda s_1 + \mu s_2$ in $\mathbb{CP}^5$, meets the Klein quadric in two real points then the congruence is hyperbolic. If the intersections are complex, the congruence is elliptic and if the line is tangent to the quadric then the congruence is parabolic. That is, the classification depends on the number of real roots of the quadratic, $s^T Q_0 s = 0$.

**Theorem 9.1.** *The example above gives a hyperbolic linear line congruence.*

*Proof.* Using the values of $s_1$ and $s_2$ given above, the quadratic,

$$(\lambda s_1 + \mu s_2)^T Q_0 (\lambda s_1 + \mu s_2) = 0$$

simplifies to,

$$\mu^2\big((n'_x)^2 + (n'_y)^2\big) - \lambda^2\big((n_x)^2 + (n_y)^2\big) = 0.$$

This has two real roots. □

Next, consider the set of lines given by group elements taking $p$ to every other point in the plane. This construction was denoted $\mathcal{L}_p$ in [4]. In [3] lines corresponding to point-pairs with a common initial point were called "left-paratactic" lines, see also [10, Theorem 8.2.19]. Referring to equation (4.1) again, and assuming $p_x$ and $p_y$ are constants then we have the conditions,

$$P_{20} - P_{13} - p_x P_{03} = 0 \quad \text{and} \quad P_{01} - P_{23} - p_y P_{03} = 0.$$

**Theorem 9.2.** *The linear congruence $\mathcal{L}_p$ is an elliptic linear congruence of lines.*

*Proof.* In this case the two linear equations can be written using,

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ -p_x \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 0 \\ 1 \\ -p_y \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

So that,

$$(\lambda s_1 + \mu s_2)^T Q_0 (\lambda s_1 + \mu s_2) = -2(\lambda^2 + \mu^2) = 0$$

This has complex roots. □

It is straightforward to see that the congruence of lines representing all point-pairs where the final point is fixed is also an elliptic linear congruence. The only change from the argument above is that the signs of the coordinates of the point changes, but this does not affect the quadratic equation that determines the type of the congruence. The congruence will be denoted $\widehat{\mathcal{L}}_q$.

*Remark* 9.3. The elliptic linear congruence has many special properties which are easily derivable. For example, no pair of lines from the congruence meet. From the interpretation of the lines as displacements taking $p$ to some other point in the plane, it is clear that two lines can't meet since this would imply that there is a displacement which moves $p$ to two different points. From this it is not too difficult to see that for every point in $\mathbb{RP}^3$ there is only one line in the congruence through the point. Similarly for each plane in $\mathbb{RP}^3$ there is only one line from the congruence which lies in that plane.

Suppose we intersect the congruence $\mathcal{L}_p$ with the quadratic complex $Q_\delta$, for some value of the separation $2\delta$. That is, we look for the set of lines in $\mathbb{RP}^3$ which correspond to moving $p$ to any point a distance $2\delta$ away. The equations that the lines must satisfy are,

$$P_{01} - P_{23} - p_y P_{03} = 0,$$
$$P_{20} - P_{13} - p_x P_{03} = 0,$$

for the congruence;

$$P_{13}^2 + P_{23}^2 - \delta^2 P_{03}^2 = 0,$$

for $Q_\delta$ and the Klein quadric;

$$P_{01}P_{23} + P_{20}P_{13} + P_{03}P_{12} = 0.$$

In $\mathbb{RP}^5$ the two linear equations determine a 3-plane, so the result will be the intersection of a pair of quadrics in an $\mathbb{RP}^3$. To look at this more closely we can use the linear equations to eliminate $P_{01}$ and $P_{20}$ from the equation for the Klein quadric. This produces,

$$(P_{23} + p_y P_{03})P_{23} + (P_{13} + p_x P_{03})P_{13} + P_{03}P_{12} =$$
$$(P_{23}^2 + P_{13}^2) + (p_x P_{13} + p_y P_{23} + P_{12})P_{03} = 0.$$

Subtracting the $Q_\delta$ quadric gives the singular quadric,

$$(\delta^2 P_{03} + p_x P_{13} + p_y P_{23} + P_{12})P_{03} = 0.$$

By construction this quadric lies in the linear system of quadrics in the 3-plane determined by the congruence. Since it factorises, it represents a pair of 2-planes. The intersection of the original quadrics is then given by the intersection of either quadric with the 2-planes, that is the result is a pair of conic curves. However, intersecting $Q_\delta$ with the plane $P_{03} = 0$ clearly gives a

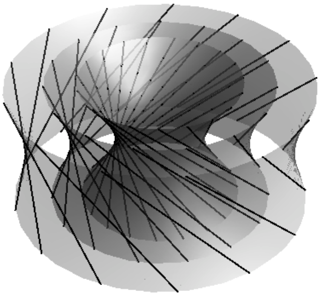

FIGURE 2. Some of the Lines in an Elliptic Linear Congruence.

complex curve. The intersection of the other 3-plane $\delta^2 P_{03} + p_x P_{13} + p_y P_{23} + P_{12} = 0$ with $Q_\delta$ gives a conic which can be written in the form,

$$(P_{13},\, P_{23},\, P_{12}) \begin{pmatrix} \delta^2 - p_x^2 & -p_x p_y & -p_x \\ -p_x p_y & \delta^2 - p_y^2 & -p_y \\ -p_x & -p_y & -1 \end{pmatrix} \begin{pmatrix} P_{13} \\ P_{23} \\ P_{12} \end{pmatrix} = 0.$$

The characteristic equation of this conic is then,

$$\det \begin{pmatrix} (\delta^2 - p_x^2) - \lambda & -p_x p_y & -p_x \\ -p_x p_y & (\delta^2 - p_y^2) - \lambda & -p_y \\ -p_x & -p_y & -1 - \lambda \end{pmatrix} = \big(\lambda^2 + (p_x^2 + p_y^2 + 1 - \delta^2)\lambda - \delta^2\big)(\delta^2 - \lambda) = 0.$$

From the fact that $\delta^2$ must be positive and the pattern of signs of the coefficients of $\lambda$ in the quadratic factor, this equation must have two positive and one negative root. Hence, the intersection is an ellipse in the Klein quadric.

In the usual geometry of the Klein quadric a conic curve represents a regulus of a hyperboloid. This gives us a nice way to visualise the elliptic linear congruence as reguli on a foliation of space by nested hyperbolas. Each hyperbola corresponds to final points of the point-pair located a distance $2\delta$ from the first point $p$, see figure 2.

These considerations lead to the natural question: How can we tell if the final point of one point-pair is the initial point of another? This is now fairly simple to answer, let the first line be $\ell_{pq}^T = (P_{13} : P_{23} : P_{12} : P_{20} : P_{01} : P_{03})$ and the second $\ell_{qr}^T = (P'_{13} : P'_{23} : P'_{12} : P'_{20} : P'_{01} : P'_{03})$. So, from equations (4.1) and (4.2), we get,

$$P_{20}P'_{03} + P_{13}P'_{03} - P_{03}P'_{20} + P_{03}P'_{13} = 0$$

and

$$P_{01}P'_{03} + P_{23}P'_{03} - P_{03}P'_{01} + P_{03}P'_{23} = 0.$$

That is, if the point-pairs satisfy the condition then these relations will be satisfied. On the other hand, it is clear that, so long as neither $P_{03}$ nor $P'_{03}$

vanish then if two lines satisfy these conditions then they will represent point-pairs in the plane where the second point of the first pair coincides with the first point of the second pair.

As an example of a congruence which is not linear, consider the congruence of lines given by requiring the initial point of a point-pair to lie on a circle with radius $r_1$ centred at the origin and final point on a radius $r_2$ circle centred at $k = (2\delta, 0)$.

**Theorem 9.4.** *The congruence of lines described by the requirements above is the intersection of the Klein quadric with another 4-dimensional quadric in* $\mathbb{RP}^5$ *and a hyperplane. That is, the intersection of a linear complex with a quadratic complex of lines.*

*Proof.* From section 8 we can see that the congruence will satisfy the two quadratic equations,

$$(P_{20} - P_{13})^2 + (P_{01} - P_{23})^2 - r_1^2 P_{03}^2 = 0,$$
$$(P_{20} + P_{13} - 2\delta P_{03})^2 + (P_{01} + P_{23})^2 - r_2^2 P_{03}^2 = 0.$$

Expanding the second equation and subtracting the first gives,

$$4P_{20}P_{13} + 4P_{01}P_{23} - 4\delta(P_{20} + P_{13})P_{03} + (4\delta^2 + r_1^2 - r_2^2)P_{03}^2 = 0.$$

Now, we can use the equation for the Klein quadric (3.4), to substitute for $P_{20}P_{13} + P_{01}P_{23})$ to produce,

$$\big((4\delta^2 + r_1^2 - r_2^2)P_{03} - 4P_{12} - 4\delta(P_{20} + P_{13})\big)P_{03} = 0.$$

The result is a quadratic equation that factorises, thus representing a quadric that consist of a pair of hyperplanes. The linear line complex given by the hyperplane $P_{03} = 0$ does not represent any point-pair in the plane. So, the congruence lies in the other linear complex and either one of the quadratic complexes. □

*Remark* 9.5. Compare the linear line complex found here with the general linear line complex studied at the end of section 7. The line complex found above is clearly a general linear complex with coefficients,

$$a_0 = -4\delta, \qquad b_0 = 0, \qquad c_0 = -4,$$
$$a_1 = -4\delta, \qquad b_1 = 0, \qquad c_1 = (4\delta^2 + r_1^2 - r_2^2).$$

From the results of section 7, we can see that the two points determined by the linear complex are $c = (c_x, c_y) = (0, 0)$ and $k = (k_x, k_y) = (2\delta, 0)$. We also have that the distance $D$ is given by,

$$D = -4\Big(\big(p_x^2 + p_y^2\big) - \big((q_x - 2\delta)^2 + q_y^2\big)\Big) = -4(r_1^2 - r_2^2).$$

So, if the initial point is restricted to a distance $r_1$ from the origin $c$, then the linear complex will constrain the final point to a circle centred at $k$ with a radius of $r_2$.

Intersecting this congruence with the quadratic line complex $Q_\delta$ from the end of section 8 will give a 1-dimensional set of lines in $\mathbb{RP}^3$ corresponding to point-pairs in the plane where the first point lies on a circle, the second lies on another circle and the points are a fixed distance apart. This is the geometry of a 4-bar mechanism, a device used and studied in Mechanical engineering. There are many different approaches to representing the configuration space of such a linkage each with its own advantages and disadvantages. This approach appears in Grünwalds original paper, [3]. Other planar 4-bar linkages could be studied using this approach, for example the elliptic trammel. For such a mechanism two points on the coupler bar lie on a pair of lines. To model this, we could take the linear congruence defined by the first point of the pair restricted to the first line and the second point lying on the second line. Intersecting this congruence with the quadratic line complex $Q_\delta$, where $2\delta$ is the distance between the points, gives the configuration space of the coupler bar. This motion is well know so we won't pursue this here.

Line congruences have a bi-degree. In general, if the bi-degree of a congruence is $(m, n)$ then $m$ is the number of intersections with a general $\alpha$-plane and $n$ is the number of intersections with a general $\beta$-plane. If the congruence is the complete intersection of the Klein quadric with a pair of hypersurfaces in $\mathbb{CP}^5$ with degrees $d_1$ and $d_2$ then the bi-degree of the congruence will be $(d_1d_2, d_1d_2)$. Hence, the line congruence $\mathcal{L}_p$ has bi-degree $(1, 1)$ since it is the complete intersection of two linear line complexes. See [13, Ch. X] or [6]. Notice that the ground field is now the complex numbers $\mathbb{C}$, this make some of the next few results simpler to state. In particular, Halphen's theorem, which gives the number of intersections between a pair of congruences. If a pair of congruences have bi-degrees $(m_1, n_1)$ and $(m_2, n_2)$ and they intersect without a common component, then the number of intersections will be $m_1m_2+n_1n_2$, where the intersections are properly counted. If we use $\mathbb{R}$ as the ground field, then this formula gives the maximum number of intersections. For example, the intersection $\mathcal{L}_p \cap \widehat{\mathcal{L}}_q$ will have $1\times 1+1\times 1 = 2$ intersections. These are easily seen to be the lines $\ell_{pq}$ and $\ell_\infty$. Although this theorem dates back to 1869 [9], it can be understood in terms of the homology of the Klein quadric.

## 10. Concluding Remarks

There are many other properties and ideas associated with lines and subspaces of lines which will have applications to point-pairs in the plane. As an example, ruled surfaces, that is, one parameter families of lines have not been considered in the above. These will correspond to one-parameter families of point-pairs, pairs of curves in other words. There will also be a 1-to-1 mapping between points on the curves, determined by the ruled surface. This might be useful for studying parallels of plane curves.

Around the end of the 19th century there was much interest in arrangements of lines in space: the 27 lines in a cubic surface, Schläfli's double six and so forth. It would interesting to see how these correspond to configurations of points in the plane.

The ideas presented here may also provide some insight into Ivory's Theorem. This is a rather old result that states that the diagonals of a curved quadrilateral formed from arcs of confocal ellipses and hyperbolas have equal length. The theorem is simple to prove given parameterisations of the ellipses and hyperbolas. However, it is known to generalised to any dimension, indeed James Ivory's original 1809 paper gave the 3-dimensional version in the context of finding the gravitational field of an ellipsoidal body, see [5].

Finally, there is another, completely different, way to associate a pair of points in the plane to a line in space. In this case we begin with circles in the plane. The set of all circles in the plane can thought of as points in an $\mathbb{RP}^3$ by using cyclographic coordinates; sometimes also called tetracyclic coordinates, see [2]. The relevant Clifford algebra for the situation is $Cl(1,3)$ or $Cl(3,1)$. The group of Möbius transformations of the plane are represented in both these algebras. More generally, in higher dimensions, this type of coordinate system is useful for studying conformal geometry. Hyperbolic pencils of circles determine a pair of points in the plane, every circle in the pencil passes through both points. Now, pencils can be thought of as lines in $\mathbb{RP}^3$, that is one dimensional linear systems. Not all lines correspond to pairs of points and the points are unordered now. If two of these lines meet this implies that the two pencils share a common circle. In other words, the four points of the two point-pairs lie on a common circle. The geometry of the Klein quadric relevant to this correspondence between point-pairs and lines in space will be different again.

## Conflict of Interests and Data Availability

There is no data associated with this work. The author states that there is no conflict of interest.

## References

[1] Bottema, O. and Roth, B., (1990), *Theoretical Kinematics*, Dover Publications, New York.

[2] J.L. Coolidge, (1916), *A Treatise on the Circle and the Sphere*, Clarendon Press, Oxford.

[3] J. Grünwald, (1911), "Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft,", *Sitzungsberichten der kaiserl. Akademie der Wissenschaften in Wein. Mathem.-naturw. Klasse*; Bd. CXX. Abt. II a, (**120**(2a): 677–741). (English translation: D. H. Delphenich, `http://www.neo-classical-physics.info/uploads/3/4/3/6/34363841/grunwald_-_kinematic_mapping.pdf`)

[4] L. Guth and N. H. Katz. (2015). "On the Erdős distinct distance problem in the plane". *Annals of Mathematics*, **181** (1): 155-190.

[5] J. Ivory, (1809), "On the Attractions of Homogeneous Ellipsoids." *Phil. Trans. R. Soc. Lond.*, pp. 345–372 .

[6] C.M. Jessop, (1903), *A treatise on the line complex.* Cambridge University Press, Cambridge.

[7] D. Klawitter and M. Hagemann, (2013), "Kinematic mappings for Cayley-Klein geometries via Clifford algebras", *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, **54** (2):737–761.

[8] B. Murphy, G. Petridis, T. Pham, M. Rudnev and S Stevens, (2022), "On the pinned distances problem in positive characteristic." *J. London Math. Soc.*, **105**(1): 469–499. https://doi.org/10.1112/jlms.12524

[9] E. Picard, (1890), "Notice sur la vie et les travaux de Georges-Henri Halphen, Membre de la Section de Géométrie", *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, **110** (N10): 489–497.

[10] H. Pottmann and J. Wallner (2010), *Computational line geometry* Mathematics and Visualization. Springer-Verlag, Berlin.

[11] M. Rudnev and J.M. Selig, (2016), "On the use of Klein quadric for geometric incidence problems in two dimensions", *SIAM J. Discrete Math.*, , **30**(2):934-954, (21 pages). doi:10.1137/16M1059412

[12] Selig J.M., 2005, *Geometrical fundamentals of robotics* (2nd ed.) Springer Verlag, New York.

[13] J.G. Semple and L. Roth, (1985), *Introduction to Algebraic Geometry*, Clarendon Press, Oxford. (First published 1949)

[14] Terence Tao, (5 March, 2011), "Lines in the Euclidean group SE(2)" https://terrytao.wordpress.com/2011/03/05/lines-in-the-euclidean-group-se2/

J.M. Selig
School of Engineering
London South Bank University
LONDON SE1 OAA
U.K.
e-mail: seligjm@lsbu.ac.uk