# Pancyclicity when each cycle must pass exactly $k$ Hamilton cycle chords

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**Abstract** It is known that $\Theta(\log n)$ chords must be added to an $n$-cycle to produce a pancyclic graph; for vertex pancyclicity, where every vertex belongs to a cycle of every length, $\Theta(n)$ chords are required. A possibly 'intermediate' variation is the following: given $k$, $1 \leq k \leq n$, how many chords must be added to ensure that there exist cycles of every possible length each of which passes exactly $k$ chords? For fixed $k$, we establish a lower bound of $\Omega(n^{1/k})$ on the growth rate.

**Keywords:** extremal graph theory, pancyclic graph, Hamilton cycle.

A simple graph $G$ on $n$ vertices is *pancyclic* if it has cycles of every length $l$, $3 \leq l \leq n$. The study of these graphs was initiated by Bondy's observation [1, 2] that, for non-bipartite graphs, sufficient conditions for hamiltonicity can also be sufficient for pancyclicity. In general, we may distinguish, in a pancyclic graph $G$, a Hamilton cycle $C$; then the remaining edges of $G$ form chords of $C$. We can then ask, given $k \leq l \leq n$ if, relative to $C$, a cycle of length $l$ exists which uses exactly $k$ chords. This suggests a $k$-chord analog of pancyclicity: do all possible cycle lengths occur when cycles must use exactly $k$-chords of a suitably chosen Hamilton cycle?

We accordingly define a function $c(n, k)$, $n \geq 6$, $k \geq 1$, to be the smallest number of chords which must be added to an $n$-cycle in order that cycles of all possible lengths may be found, each passing exactly $k$ chords. No Hamilton cycle can use exactly one chord of another Hamilton cycle, so that when $k = 1$ cycle lengths must lie between $k$ and $n - 1$. The function is undefined for $k > n$. We define the function for $n \geq 6$ because $n = 4, 5$ are too restrictive to be of interest to us.

Our aim in this paper is to investigate the growth of the function $c(n, k)$ as $n$ increases, for fixed $k$.

**Example 1** *Label the vertices around the cycle $C_6$, in order, as $v_1, \ldots, v_6$. Add chords $v_1 v_3$ and $v_1 v_4$; the result is a pancyclic graph. It also has cycles of all lengths $\leq 5$ each passing exactly one of the chords. If $v_2 v_6$ is added then cycles exist of all lengths $\geq 3$, each passing two chords. If two further chords, $v_2 v_4$ and $v_4 v_6$, are added then cycles exist of all lengths $\geq 3$, each passing three chords. For 4-chord cycles we require six chords to be added, i.e. $C(6, 4) = 6$. Six suitably chosen chords are also sufficient for $5 - chord$ and $6 - chord$ cycles: $C(6, 5) = C(6, 6) = 6$.*

**Lemma 2**    *(1)* $c(n, 1) = \left\lfloor \dfrac{n - 3}{2} \right\rfloor.$

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*(2)* $c(n, k) \geq k$, *with equality if and only if* $k = n$.

*(3)* $c(n, n - 1) = n$.

**Proof.** (1) follows from the observation that a chord in $C_n$ forming a 1-chord cycle of length $k$ automatically forms a 1-chord cycle of length $n + 2 - k$.

(2) is immediate from the definition of $c(n, k)$.

(3) Let $G$ consist of an $(n - 1)$-cycle, together with an $(n - 1)$-chord cycle on the same vertices. Choose vertex $v$: let the chords at $v$ be $xv$ and $yv$ and its adjacent cycle edges be $uv$ and $vw$, with $u, v, w, x, y$ appearing in clockwise order around the cycle. Replace $v$ and its incident edges with two vertices $v_u$ and $v_w$, with edges $v_u v_w$, $u v_u$, $v_w w$, $x v_w$ and $y v_u$. The $(n - 1)$-chord cycle in $G$ becomes an $(n - 1)$-chord $n$-cycle. Add an $n$-th chord $x v_u$ to give an $(n - 1)$-chord $(n - 1)$-cycle. $\square$

Table 1 supplies some small values/bounds for $c(n, k)$. The lower bounds are supplied by Corollary 7 (see below); except for those values covered by Lemma 2, exact values and upper bounds were found by computer search.

| | | $k$ | | | | | | | | | | |
| | | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | **6** | 2 | 3 | 5 | 6 | 6 | 6 | | | | | |
| | **7** | 2 | 3 | 5 | 6 | 6 | 7 | 7 | | | | |
| | **8** | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 8 | | | |
| | **9** | 3 | 4 | 5 | 6 | 7 | 8 | 8 | 9 | 9 | | |
| | **10** | 4 | 4 | 5 | 6 | $\geq 6$ | $\geq 7$ | $\geq 8$ | $\geq 9$ | 10 | 10 | |
| | **11** | 4 | 4 | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 7$ | $\geq 8$ | $\geq 9$ | $\geq 10$ | 11 | 11 |
| | **12** | 5 | 4 | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 7$ | $\geq 8$ | $\geq 9$ | $\geq 10$ | $\geq 11$ | 12 |
| | **13** | 5 | 4 | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 8$ | $\geq 8$ | $\geq 9$ | $\geq 10$ | $\geq 11$ | $\geq 12$ |

Table 1. Values of $c(n, k)$ for $6 \leq n \leq 13$ and $1 \leq k \leq 11$.

Our aim is to compare $c(n, k)$ with the number of chords required for pancyclicity and for *vertex pancyclicity*, in which each vertex must lie on a cycle of every length.

The following lower bound is stated without proof in [1]:

**Theorem 3** *In a pancyclic graph $G$ on n vertices the number of edges is not less than $n - 1 + log_2(n - 1)$.* $\square$

For the sake of completeness we observe that theorem 3 follows immediately from the following lemma:

**Lemma 4** *Suppose p chords are added to $C_n$, $n \geq 3$. Then the number $N(n, p)$ of cycles in the resulting graph satisfies*

$$\binom{p + 2}{2} \leq N(n, p) \leq 2^{p+1} - 1.$$

**Proof.** Embed $C_n$ convexly in the plane. Suppose the chords added to $C_n$ are, in order of inclusion, $e_1, e_2, \ldots, e_p$. Say that $e_i$ intersects $e_j$ if these edges cross each other when added to

the embedding of $C_n$. Let $n_i$ be the number of new cycles obtained with $e_i$ is added. Then $n_i$ satisfies:

1. $n_i \geq i + 1$, the minimum occurring if and only if the $e_j$ are pairwise non-intersecting for $j \leq i$;

2. $n_i \leq 2^i$, the maximum occurring if and only if $e_i$ intersects with $e_j$ for all $j < i$, giving $n_i = \sum_{j=0}^{i} \binom{i}{j}$.

Now $1 + \sum_{i=1}^{p} (i + 1) \leq 1 + \sum_{i=1}^{p} n_i \leq 1 + \sum_{i=1}^{p} 2^i$ and the result follows. $\qquad\square$

The exact value of the minimum number of edges in an $n$-vertex pancyclic graph has been calculated for small $n$ by George et al [5] and Griffin [6]. For $3 \leq n \leq 14$ the lower bound in theorem 3 is exact; however, it can be seen that, for $n = 15, 16$, we must add four chords to $C_n$ to achieve pancyclicity while the argument in the proof of lemma 4 can only account for three.

As regards an upper bound on the number of chords required for pancyclicity, [1] again asserts $O(\log n)$, again without a proof. A $\log n$ construction has been given by Sridharan [7]. Together with theorem 3 this gives an 'exact' growth rate for pancyclicity: it is achieved by adding $\Theta(\log n)$ chords to $C_n$.

In contrast, *vertex pancyclicity*, in which every vertex lies in a cycle of every length has been shown by Broersma [3] to require $\Theta(n)$ edges to be added to $C_n$. Our question is: where between $\log n$ and $n$ does $c(n, k)$ lie? For fixed $k$, we find a lower bound strictly between the two: $\Omega(n^{1/k})$.

Let us for the moment restrict to $k \geq 3$. Suppose we add $p$ chords to $C_n$, $3 \leq k \leq p \leq \binom{n}{2} - n$. Suppose that these $p$ added chords include a $k$-cycle. We will use $K(k, p)$, defined for $k \geq 3$, to denote the maximum number of $k$-chord cycles that can be created in the resulting graph. Then $1 \leq K(k, p)$ by definition and $K(k, p) \leq 2^{p+1} - 1$ by lemma 4. By lowering this upper bound we can increase the lower bound on C(n,k).

**Theorem 5** $K(k, p) \leq \binom{p}{k} + k\binom{p - k}{k - 1} + \binom{p - k}{k}$.

We will use the following Lemma to prove theorem 5:

**Lemma 6** *Suppose that a set $X$ of chords is added to $C_n$. In the resulting graph the maximum number of cycles passing all edges in $X$ is*

$$
\begin{cases}
1 & \text{if } X \text{ contains adjacent chords} \\
2 & \text{if no two chords of } X \text{ are adjacent}
\end{cases}
$$

**Proof.** Let $G$ be the graph resulting from adding the chords of $X$ to $C_n$. We may assume without loss of generality that $G$ has no vertices of degree 2, since such vertices may be contracted out. For a given cycle in $G$ passing all chords of $X$, let $H$ denote the intersection of this cycle with the $C_n$. Then $H$ consists of isolated vertices and disjoint edges, and $H$ is completely determined once any of these vertices or edges is fixed. If two chords are adjacent this fixes an isolated vertex of $H$; if no two chords are adjacent then there is a maximum of two ways in which a single edge of $H$ may be fixed. $\qquad\square$

**Proof of theorem 5.** By definition of $K(k, p)$ we must use a set, say $S$, of $k$ chords to create a $k$-cycle. We add new chords to $S$, one by one. On adding the $r$-th additional chord, $1 \le r \le p-k$, we ask how many $k$-chord cycles use this chord. For any such a cycle the previous $r-1$ chords will be split between $S$ and non-$S$ chords: with $i$ chords from $S$ being used, $0 \le i \le k-1$, this can happen in

$$\binom{k}{i}\binom{r-1}{k-i-1}$$

ways. Since $i > 1$ forces two adjacent chords in $S$ to be used, summing over $i$, according to lemma 6, and then over $r$ gives

$$K(k, p) \le 1 + \sum_{r=1}^{p-k}\left(2\sum_{i=0}^{1}\binom{k}{i}\binom{r-1}{k-i-1} + \sum_{i=2}^{k-1}\binom{k}{i}\binom{r-1}{k-i-1}\right).$$

This simplifies (e.g. using symbolic algebra software such as Maple) to give the result. □

**Corollary 7** *For given positive integers $k$ and $n$, with $3 \le k \le n$ and $n \ge 6$, the value of $c(n, k)$ is not less than the largest root of the following polynomial in $p$:*

$$\Pi(p; n, k) = \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} - n + k - 1.$$

□

We finally extend our analysis to include the cases $k = 1, 2$:

**Corollary 8** *Let $n \ge 6$ be a positive integer. Then for $k \ge 1$ fixed, $c(n, k)$ is of order $\Omega(n^{1/k})$.*

**Proof.** For $k = 1$ the required linear bound was provided in lemma 2.

For $k = 2$ an analysis similar to that used in the proof of theorem 5 shows that the number of 2-chord cycles which may be created by adding $p$ chords to $C_n$ is at most $p^2 - p - 1$. So to have 2-chord cycles of all lengths from 3 to $n$ we require $p^2 - p - 1 \ge n - 2$. In this case we can solve explicitly to get the bound $p \ge \frac{1}{2}\left(1 + \sqrt{4n - 3}\right)$.

Now suppose $k \ge 3$. In order to have all $k$-chord cycles of all lengths between $k$ and $n$ we must have

$$n - k + 1 \le \binom{p}{k} + k\binom{p-k}{k-1} + \binom{p-k}{k} \le f(k)p^k,$$

for some function $f(k)$. Therefore $p^k \ge (n - k + 1)/f(k)$ so, for $k$ fixed, $p = \Omega(n^{1/k})$. □

**Remarks 9**  *1. We are suggesting that the value of $c(n, k)$ may be 'intermediate' between pancyclicity and vertex pancyclicity in the sense that the number of chords it requires to be added to $C_n$ may lie between $\log n$ and $n$. Thus far we have only a lower bound in support of our suggestion. Moreover, a comparison of the growth orders, $\Omega(\log n)$ as opposed to $\Omega(n^{1/k})$, suggests that this is very much a 'for large n' type result. The equation $\ln n = n^{1/k}$ has two positive real solutions for $k \ge 3$, given in terms of the two real branches of the Lambert W function [4]. In particular $\ln n$ exceeds $n^{1/k}$ for $n > e^{-kW_{-1}(-1/k)}$, and this bound grows very fast with $k$: at least two orders of magnitude per unit increase! To give a specific example, $k = 10$, the $\log$ bound exceeds the 10-th root bound until the number of vertices exceeds about $3.4 \times 10^{15}$. Until then, so far as our analysis goes, we might expect 'most' pancyclic graphs to be 10-chord pancyclic. However we suggest that, in the long term, a guarantee of this implication, analogous to hamiltonicity guaranteeing pancyclicity, will not be found.*

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2. *We would like to know if $c(n, k)$ is monotonically increasing in n. However, it is still open even whether pancyclicity is monotonic in the number of chords requiring to be added to $C_n$ (the question is investigated in [6]). We believe that $c(n, k)$ it is not increasing in k and $c(n, 1) > c(n, 2)$ for $n = 12, 13$ confirms this in a limited sense. Our $n^{1/k}$ lower bound instead suggests the possibility that $c(n, k)$ is convex for fixed n, as a function of k.*

3. *We observe that, unlike pancyclicity, the property of having cycles of all lengths each passing k chords is not an invariant of a graph: it depends on the initial choice of a Hamilton cycle. For example, in figure 1, there are cycles of all lengths $\leq 9$ each passing exactly one of the $c(10, 1) = 4$ chords of the outer cycle but there is no 4-cycle passing exactly one chord of the bold-edge Hamilton cycle.*
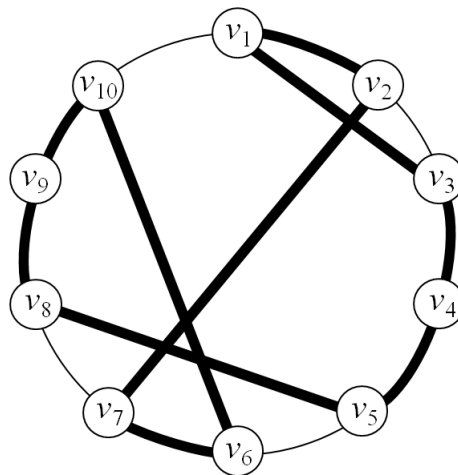


Figure 1: No 4-cycle uses exactly 1 chord of the bold-edge Hamilton cycle.

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