Equimomental Systems and Robot Dynamics

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Abstract

Three different concepts from the past are reviewed from a more modern standpoint. Constructing an equimomental system of four point-masses to an arbitrary rigid-body; the conditions for the generalised mass matrix of a serial robot to be constant and how to dynamically balance an arbitrary rigid body so that it is symmetrical about a given axis. Connections are made to the geometry of a 3-dimensional Veronese variety.

1. Introduction

It is well known that two rigid bodies can have the same dynamical properties. Two rigid bodies or systems of rigidly connected point-masses which have the same dynamical properties are said to be *equimomental*. The condition for systems to be equimomental is that their centres of mass coincide and that their moments and products of inertia are equal in some fixed coordinate system. These ideas seem to date back to Routh (1950), who showed that for any lamina there is always an equimomental system of three point masses and that for 3-dimensional bodies an equimomental system of four point masses can always be found. A short proof of this appears in Sommerville Sommerville (1930) and also Huang (1993). Chaudhary & Saha (2009) have used these ideas more recently for mechanism balancing applications.

Here the result that every rigid body is equimomental to a system of four point masses is reviewed using modern ideas and notation. This allows us to say a little more about the possible solutions to the problem. Next the dynamics of serial robot arms is studied with the purpose of trying to simplify the robot's equations of motion. The possible simplification considered is how the generalised mass matrix of the robot can be made constant. This was first studied by Asada & Youcef-Toumi (1986) who showed that the inertias of each link must be symmetrical with respect to the corresponding joint axis.

In the final section the problem of adding counterbalancing weights to an arbitrary rigid-body to make it symmetrical with respect to an arbitrary axis is considered. This is another classical problem, the problem of dynamically balancing a rotor.

The first section looks at the inertia matrix of point masses and of rigid bodies in general.

2. The Inertia Matrix

For rigid-body dynamics it is often useful to use a six-dimensional formalism to describe the position and orientation of the body, see for example Selig (2005). In such a formalism the inertia of the body is represented as a 6×6 matrix with the general form,

$$N = m \begin{pmatrix} \mathbb{I} & C \\ C^T & I_3 \end{pmatrix},$$

where m is the mass of the body; N is usual 3×3 inertia matrix of the body (divided by m); I_3 is the 3×3 identity matrix and C is an anti-symmetric matrix corresponding to the position vector of the centre of mass. Suppose $\mathbf{c} = (c_x,\, c_y,\, c_z)^T$ is the position vector of the centre of mass then C is defined by requiring $C\mathbf{p} = \mathbf{c} \times \mathbf{p}$ for arbitrary position vectors $\mathbf{p}$.

In this formalism the velocity of the rigid body is represented by a 6-vector which can be partitioned into a pair of 3-vectors,

$$\mathbf{s} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix},$$

with $\boldsymbol{\omega}$ the angular velocity of the body and $\mathbf{v}$ the linear velocity of the origin of coordinates as a point on the body. With this formalism the kinetic energy of the body is given by $\mathrm{KE} = (1/2)\mathbf{s}^T N \mathbf{s}$. The generalised velocity vectors $\mathbf{s}$, are usually referred to as twists.

In the present work a slightly different formalism will be used. The independent entries in inertia matrix N can be arranged as a symmetric 4×4 matrix,

$$\widetilde{\Xi} = m \begin{pmatrix} \frac{1}{2}(-I_{xx} + I_{yy} + I_{zz}) & -I_{xy} & -I_{xz} & c_x \\ -I_{xy} & \frac{1}{2}(I_{xx} - I_{yy} + I_{zz}) & -I_{yz} & c_y \\ -I_{xz} & -I_{yz} & \frac{1}{2}(I_{xx} + I_{yy} - I_{zz}) & c_z \\ c_x & c_y & c_z & 1 \end{pmatrix},$$

Selig & Martin (2014) referred to this as the "homogeneous plane-distance inertia matrix". Here, for brevity, the matrix will be referred to as the 4×4 inertia matrix of the body. The entries I_{ij} refer to the corresponding entries of the 3×3 inertia matrix. This can be thought of as specifying a linear mapping from the 6×6 inertia matrices to the space of homogeneous plane-distance inertia matrices, moreover it is clear that the map is invertible. Hence we may assume that two rigid bodies are equimomental if and only if their homogeneous plane-distance inertia matrices are the same.

The 4×4 inertia matrix of a point mass located at position vector $\mathbf{p}$ and with mass m can be written as the product,

$$\widetilde{\Xi} = m\tilde{\mathbf{p}}\tilde{\mathbf{p}}^T.$$

In this expression $\tilde{\mathbf{p}}$, will be referred to as an extended position vector and has the form, $\tilde{\mathbf{p}} = (p_x,\, p_y,\, p_z,\, 1)^T$. The expression above is a simple consequence of the definitions of the inertia matrix. Notice that this also tells us how the 4×4 inertia matrix transforms under a rigid body displacement, the extended vectors clearly transform according to the standard homogeneous representation of $SE(3)$ and hence the inertia matrix will transform according to,

$$\widetilde{\Xi}' = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \widetilde{\Xi} \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}^T,$$

where R is the $3\times$ rotation matrix of the displacement and $\mathbf{t}$ the translation vector.

It is a classical theorem that for any rigid body there is a translation which will position the centre of mass at the origin and a rotation that will align the principal directions of inertia with the coordinate axes. Such a rigid displacement will hence diagonalise the 6×6 inertia matrix. It is clear that the same rigid displacement will also diagonalise the 4×4 inertia matrix. If the principal moments of inertia are mk_1^2, mk_2^2 and mk_3^2 then the diagonal entries of the 4×4 inertia matrix will be,

$$ma^2 = \frac{m}{2}(-k_1^2 + k_2^2 + k_3^2), \quad mb^2 = \frac{m}{2}(k_1^2 - k_2^2 + k_3^2), \quad mc^2 = \frac{m}{2}(k_1^2 + k_2^2 - k_3^2).$$

These will be positive by the triangle inequality satisfied by the principal moments of inertia.

Using this representation we can think of the space of all possible inertia matrices as points in a projective space $\mathbb{P}^9$, with homogeneous coordinates given by the 10 independent entries of $\tilde{\Xi}$. This introduces some unphysical points, in particular the hyperplane determined by $\tilde{\xi}_{44} = 0$. It should also be kept in mind that no distinction between positive definite and non-positive definite matrices has been made.

The space of all point-masses can be viewed as the image of the quadratic Veronese map from $\mathbb{P}^3$ to $\mathbb{P}^9$. The image is a 3-dimensional variety of degree $2^3 = 8$, Harris (1992). It is determined by a number of quadratic equations, these are the equations which express the fact that $\tilde{\Xi}$ has rank 1.

3. Four Point-Masses

In this section the theorem due to Routh (1950) will be reviewed.

Consider 4 point-masses with equal mass, arranged at the vertices of a regular tetrahedron. By choosing coordinates with the origin at the centre of mass, the z-axis aligned with the position vector of the first point and the xz-plane defined by the second point, the position vectors of the four point can be written as,

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} \frac{2\sqrt{2}}{3} \\ 0 \\ \frac{-1}{3} \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} \frac{-\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} \\ \frac{-1}{3} \end{pmatrix}, \quad \mathbf{p}_4 = \begin{pmatrix} \frac{-\sqrt{2}}{3} \\ \frac{-\sqrt{6}}{3} \\ \frac{-1}{3} \end{pmatrix}.$$

Now this tetrahedron can be scaled by $\sqrt{3}$ units and embedded in $\mathbb{R}^4$. This give extended position vectors,

$$\tilde{\mathbf{p}}_1 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_2 = \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_3 = \begin{pmatrix} \frac{-\sqrt{2}}{\sqrt{3}} \\ \sqrt{2} \\ \frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}, \quad \tilde{\mathbf{p}}_4 = \begin{pmatrix} \frac{-\sqrt{2}}{\sqrt{3}} \\ -\sqrt{2} \\ \frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}.$$

Notice that these extended vectors satisfy the relations, $\tilde{\mathbf{p}}_i^T \tilde{\mathbf{p}}_j = 0$ when $i \neq j$, and $\tilde{\mathbf{p}}_i^T \tilde{\mathbf{p}}_i = 4$ for $i = 1, \ldots, 4$.

If these 4 points all have masses $m/4$ then the 4×4 inertia matrix of the system will be,

$$\tilde{\Xi} = \frac{m}{4} \sum_{i=1}^{4} \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_i^T = m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

That is m times the 4×4 identity matrix.

Above we saw that for an arbitrary rigid body there is always a rigid change of coordinates that will make the 4×4 inertia matrix diagonal; $\tilde{\Xi} = m\,\mathrm{diag}(a^2, b^2, c^2, 1)$. So the regular tetrahedral system of point masses can be subject to a non-rigid similarity transformation; $D = \mathrm{diag}(a, b, c, 1)$, and the system of point masses will have the same 4×4 inertia matrix as the rigid body. That is,

$$\tilde{\Xi} = mD^T I_4 D = \frac{m}{4} \sum_{i=1}^{4} D^T \tilde{\mathbf{p}}_i \tilde{\mathbf{p}}_i^T D = \frac{m}{4} \sum_{i=1}^{4} \tilde{\mathbf{p}}'_i (\tilde{\mathbf{p}}'_i)^T. \tag{3.1}$$

The extended position vectors of the points will be given by $\tilde{\mathbf{p}}_i' = D^T\tilde{\mathbf{p}}_i$. So, for example,

$$\tilde{\mathbf{p}}_4' = \begin{pmatrix} \frac{-a\sqrt{2}}{\sqrt{3}} \\ -b\sqrt{2} \\ c\frac{-1}{\sqrt{3}} \\ 1 \end{pmatrix}.$$

The above exposition allows us to say a little more about this problem. We seek other solutions to the problem, are there other sets of four points equimomental to the original body? Suppose that in equation (3.1) above we had used UD rather than just D, where $U \in O(4)$ is an orthogonal 4×4 matrix. The 4×4 inertia matrix would not have been affected but the extended points would now be given by, $\tilde{\mathbf{p}}_i' = D^T U^T \tilde{\mathbf{p}}_i$. Since $O(4)$ is a six dimensional Lie group this gives us a six parameter family of solutions. This family is not isomorphic to $O(4)$ since the points are unordered, any permutation of the four points will give the same system, so in fact the family of solution will be isomorphic to the quotient $O(4)/S_4$ where S_4 denotes the symmetric group on 4 letters.

In terms of the geometry of the Veronese variety introduced in section 2, the above shows that there is a six parameter family of secant 3-planes through any point in $\mathbb{P}^9$.

Classically the points $\tilde{\mathbf{p}}_i' = (x,\, y,\, z,\, 1)^T$ are known to lie on the *equimomental ellipsoid*. If $\tilde{\mathbf{p}}_i' = D^T U^T \tilde{\mathbf{p}}_i$ then $\tilde{\mathbf{p}}_i = UD^{-T}\tilde{\mathbf{p}}_i'$ and hence,

$$(\tilde{\mathbf{p}}_i')^T D^{-1} U^T U D^{-T} \tilde{\mathbf{p}}_i' = 4,$$

since $\tilde{\mathbf{p}}_i^T \tilde{\mathbf{p}}_i = 4$ and $U^T U = I_4$. So substituting for $\tilde{\mathbf{p}}_i'$ gives the equation of the equimomental ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$$

Notice that the problem of finding four rank one 4×4 symmetric matrices which sum to the given full rank matrix $\tilde{\Xi}$ is a standard problem in linear algebra. The standard solution would be to find the eigenvalues λ_i, and eigenvectors $\tilde{\mathbf{e}}_i$, of the matrix. For the diagonal matrix $\tilde{\Xi} = \mathrm{diag}(a^2,\, b^2,\, c^2,\, 1)$, we would get, $\tilde{\Xi} = \lambda_1\tilde{\mathbf{e}}_1\tilde{\mathbf{e}}_1^T + \lambda_2\tilde{\mathbf{e}}_2\tilde{\mathbf{e}}_2^T + \lambda_3\tilde{\mathbf{e}}_3\tilde{\mathbf{e}}_3^T + \lambda_4\tilde{\mathbf{e}}_4\tilde{\mathbf{e}}_4^T$, where $\lambda_1 = a^2$, $\lambda_2 = b^2$, $\lambda_3 = c^2$ and $\lambda_4 = 1$. The eigenvectors are given by,

$$\tilde{\mathbf{e}}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \tilde{\mathbf{e}}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

However, this solution is not physical since the first three points lie on the plane at infinity. On the other hand, the extended vectors $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_4$ form an orthonormal frame for $\mathbb{R}^4$ as do the extended vectors $\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_4$ given above, (after scaling by 1/4). It is well known that the group $O(4)$ acts transitively on the set of orthonormal frames and hence this standard solution by eigenvectors (suitably scaled) lies in the 6-parameter family of solutions described above.

4. Constant Mass Matrix for Serial Robots

In the absence of gravity the dynamics of a serial robot arm is often summarised as,

$$M_{ij}\ddot{\theta}_j + C_{ijk}\dot{\theta}_j\dot{\theta}_k = \tau_i.$$

Here summation is intended over repeated indices, for a six joint robot the range of the indices are $1, \ldots, 6$. The variable θ_i denotes the angle of the ith joint and τ_i is the torque applied to the ith joint. The matrix M_{ij} is called the generalised mass matrix of the system and the tensorC_{ijk} contains the Coriolis and interaction terms.

In general, for a 6-joint robot, it can be shown that the generalised mass matrix has entries of the form,

$$M_{ij} = \begin{cases} \mathbf{s}_i^T(N_i + \cdots + N_6)\mathbf{s}_j, & \text{if } i \geq j, \\ \mathbf{s}_j^T(N_j + \cdots + N_6)\mathbf{s}_i, & \text{if } i > j, \end{cases}$$

where $\mathbf{s}_i$ are the twists corresponding to axis of the ith joint and N_j is the 6×6 inertia matrix of the jth link, see Selig (2005).

Several attempts have been made to design robots in such a way that the mass matrix is simplified. For example in Selig (2005) and more completely in Selig & Martin (2014), the problem of placing the joint twists $\mathbf{s}_i$ in such a way that the mass matrix becomes diagonal was studied.

Here the simpler requirement that the mass matrix remain constant is studied. This idea seem to date back to Asada & Youcef-Toumi (1986) but was also studied by Stokes & Brockett (1996). The advantage of this scheme is that the tensor C_{ijk} will disappear from the equations of motion since it is essentially a matrix of derivatives of the entries of the mass matrix. Thus the equations of motion for the robot will be greatly simplified and the control problem will be easier. The disadvantage is that as soon as the robot picks up a payload the conditions for balance will be destroyed. However, the effects of a payload are quite simple to compute.

To begin consider a simple machine with just two joints and links, its mass matrix is,

$$M = \begin{pmatrix} \mathbf{s}_1^T(N_1 + N_2)\mathbf{s}_1 & \mathbf{s}_1^T N_2 \mathbf{s}_2 \\ \mathbf{s}_2^T N_2 \mathbf{s}_1 & \mathbf{s}_2^T N_2 \mathbf{s}_2 \end{pmatrix}.$$

These quantities are to be calculated in a fixed coordinate system, hence as the robot moves they will change. For simplicity both joints will be assumed to be revolutes. The quantities $\mathbf{s}_1^T N_1 \mathbf{s}_1$ and $\mathbf{s}_2^T N_2 \mathbf{s}_2$ are already constant and need not be considered further. The term $\mathbf{s}_1^T N_2 \mathbf{s}_1$ will not be effected by a rotation about the first joint. The effect of rotation about the second joint can be written using exponentials,

$$\mathbf{s}_1^T e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)^T} N_2 e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} \mathbf{s}_1.$$

Similarly the off-diagonal term will vary according to,

$$M_{12}(\theta_1, \theta_2) = \mathbf{s}_1^T e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)^T} N_2 \mathbf{s}_2 = \mathbf{s}_1^T e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)^T} N_2 e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} \mathbf{s}_2,$$

the last equality in the above is because the joint twist $\mathbf{s}_2$ is invariant under rotations about itself. In both cases the terms will be constant if the inertia of the second link N_2, is invariant with respect to rotations about the second joint,

$$e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)^T} N_2 e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} = N_2,$$

or with a slight rearrangement,

$$e^{-\theta_2 \,\mathrm{ad}(\mathbf{s}_2)^T} N_2 - N_2 e^{\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} = 0. \tag{4.1}$$

Using Rodrigues' formula to expand the exponential gives,

$$e^{\theta_2 \,\mathrm{ad}(\mathbf{s}_2)} = I_6 + \sin\theta_2 \,\mathrm{ad}(\mathbf{s}_2) + (1 - \cos\theta_2)\,\mathrm{ad}^2(\mathbf{s}_2).$$

This formula is only valid because $\mathbf{s}_2$ is a pure rotation, for a helical joint the expression for the exponential would have more terms but the conclusion below would be the same. Substituting the Rodrigues formula into equation (4.1) and comparing the coefficients of

$\sin\theta_2$ and $\cos\theta_2$ separately gives two equations,

$$\mathrm{ad}(\mathbf{s}_2)^T N_2 + N_2\,\mathrm{ad}(\mathbf{s}_2) = 0,$$
$$(\mathrm{ad}^2(\mathbf{s}_2))^T N_2 - N_2(\mathrm{ad}^2(\mathbf{s}_2)) = 0.$$

If the first of these equations is satisfied the second will be satisfied automatically. So the condition for the mass matrix to be constant is, $\mathrm{ad}(\mathbf{s}_2)^T N_2 + N_2\,\mathrm{ad}(\mathbf{s}_2) = 0$.

Notice that this condition is simply $\partial N_2/\partial\theta_2 = 0$ evaluated at $\theta_2 = 0$. Hence, in terms of the 4×4 inertia matrix, the condition is, $S_2\widetilde{\Xi}_2 + \widetilde{\Xi}_2 S_2^T = 0$, where S_2 is the 4×4 matrix representing the Lie algebra element $\mathbf{s}_2$. That is,

$$S_2 = \begin{pmatrix} \Omega_2 & \mathbf{v}_2 \\ 0 & 0 \end{pmatrix},$$

with Ω_2 the 3×3 anti-symmetric matrix for the direction of the rotation axis and $\mathbf{v}_2$ the moment of the axis. The right-hand column of this condition gives,

$$\Omega_2\mathbf{c} + \mathbf{v}_2 = \boldsymbol{\omega}_2 \times \mathbf{c} + \mathbf{v}_2 = 0.$$

This implies that the centre of mass $\mathbf{c}$, lies on the axis of the joint. Place the origin of coordinates at the centre of mass so $\mathbf{c} = \mathbf{v}_2 = \mathbf{0}$; and align the z-axis with the direction of the joint, so $\boldsymbol{\omega}_2 = (0,\,0,\,1)^T$. Now the condition reads,

$$\Omega_2\Xi + \Xi\Omega_2 = 0.$$

Expanding this equation in terms of the elements of the inertia matrix gives,

$$\begin{pmatrix} -2\xi_{12} & \xi_{11}-\xi_{22} & \xi_{23} \\ \xi_{11}-\xi_{22} & 2\xi_{12} & \xi_{13} \\ -\xi_{23} & \xi_{13} & 0 \end{pmatrix} = 0.$$

This implies that the matrix Ξ, and hence $\widetilde{\Xi}$ is diagonal, since the off-diagonal elements $\xi_{12} = \xi_{13} = \xi_{23} = 0$. Also the joint must lie along one of the principal axes of the inertia matrix. Further we must have that $\xi_{11} = \xi_{22}$, this implies that two of the principal moments of inertia are equal.

Given a rigid body with inertia matrix N_2 and a line $\mathbf{s}_2$ let us say that the body is symmetrical about the line if the line passes through the body's centre of mass parallel to the direction of one of the principal axes of inertia and the moments of inertia associated with the two perpendicular principal axes are equal. This allows us to state succinctly the result just derived: The mass matrix of a 2-joint robot will be constant if the second link is symmetrical about the second joint axis. For robots with more than 2 links and joints this condition can be applied recursively, the final link must be symmetrical about the final joint, then the composite body consisting of the last two links must be symmetrical about the last-but-one joint and so forth.

For a particular axis, the set of inertia matrices $\widetilde{\Xi}$ which are symmetric with respect to this axis form a 3-plane in $\mathbb{P}^9$, since the ten independent entries of these matrices satisfy six linear equations. This 3-plane meets the Veronese variety described above, in a conic curve. To see this recall that the points in the Veronese variety are inertia matrices of rank 1, $\widetilde{\Xi}_p = m\tilde{\mathbf{p}}\tilde{\mathbf{p}}^T$. The equation $S_2\widetilde{\Xi}_p + \widetilde{\Xi}_p S_2^T = 0$, is solved if $S_2\tilde{\mathbf{p}} = 0$ which implies that the point $\tilde{\mathbf{p}}$ lies on the axis S_2. The solution is a line in $\mathbb{P}^3$ but the Veronese embedding has degree 2 and so the image of the line will be a conic.

5. Dynamic Balancing

Next we look at the problem of adding balancing weights, (point-masses), to an arbitrary rigid body so that the inertia matrix of the composite body is symmetric relative to a given axis. These ideas are very similar to the classical ideas of dynamic balancing of rotating machinery.

THEOREM 1. *Any rigid-body can be balanced by adding two point-masses at suitable locations.*

Proof. Choose coordinates in such a way that the given axis is the z-axis of coordinates. Let the two point masses have position vectors $\mathbf{p}_1 = (x_1, y_1, z_1)^T$ and $\mathbf{p}_2 = (x_2, y_2, z_2)^T$ with masses m_1 and m_2 respectively. The equations to solve for dynamic balance about the z-axis are,

$$m\xi_{11} + m_1x_1^2 + m_2x_2^2 = m\xi_{22} + m_1y_1^2 + m_2y_2^2, \tag{5.1}$$

$$m\xi_{12} + m_1x_1y_1 + m_2x_2y_2 = 0, \tag{5.2}$$

$$m\xi_{13} + m_1x_1z_1 + m_2x_2z_2 = 0, \tag{5.3}$$

$$m\xi_{23} + m_1y_1z_1 + m_2y_2z_2 = 0, \tag{5.4}$$

$$mc_x + m_1x_1 + m_2x_2 = 0, \tag{5.5}$$

$$mc_y + m_1y_1 + m_2y_2 = 0. \tag{5.6}$$

These six equations can be made multi-homogeneous by including a homogenising variable w into terms with a ξ_{ij} or a c_α with the appropriate degree. We also need to assume m, the mass of the original body, is variable. Rather we take m_1/m and m_2/m as our variables. In this way the equations become multi-homogeneous in the $\mathbb{P}^2 \times \mathbb{P}^6$ with multi-homogeneous coordinates $(m_1 : m_2 : m) \times (x_1 : y_1 : z_1 : x_2 : y_2 : z_2 : w)$. The six equations will form a 2-dimensional subvariety in this 8-dimensional ambient space, hence solutions will always exist. □

The usual caveats about real solutions and solutions at infinity ($\{w = 0\} \cup \{m = 0\}$), will of course, apply. It may also happen here that the masses of the counter weights turn out to be negative. For some bodies such a point-mass can be implemented by drilling a hole in the body.

This simple theorem has a couple of useful consequences, space only allow one to be considered. When balancing rotating machinery it is usual to fix the planes where the counter-weights can be fixed. This can be thought of as fixing the variables z_1 and z_2.

COROLLARY 1. *Consider a fixed axis* $\mathbf{s}$ *and a pair of distinct planes perpendicular to this axis. For an arbitrary rigid body there will be a unique way to balance the body with respect to* $\mathbf{s}$ *by adding two point-masses, one on each of the planes.*

Proof. As usual we will choose coordinates so that the fixed axis is the z-axis. In this way the two planes have the equations $z = z_1$ and $z = z_2$, for some constants z_1, z_2. The last for equations (5.3)–(5.6) are now linear equations in the variables m_1x_1, m_1y_1, m_2x_2 and m_2y_2. The solutions are,

$$m_1x_1 = m\frac{\xi_{13} - c_xz_2}{z_2 - z_1}, \quad m_1y_1 = m\frac{\xi_{23} - c_yz_2}{z_2 - z_1},$$

$$m_2x_2 = -m\frac{\xi_{13} - c_xz_1}{z_2 - z_1}, \quad m_2y_2 = -m\frac{\xi_{23} - c_yz_1}{z_2 - z_1}.$$

Substituting into equation (5.1) gives a linear equation in the variables x_1, y_1, x_2, y_2. The above results can be substituted into equation (5.2) in four different ways, since each of the two terms such as $m_1x_1y_1$ can be written as either $(m_1x_1)y_1$ or $(m_1y_1)x_1$. Each of the possible substitutions will produce a linear equation in the variables x_1, y_1, x_2, y_2. At least one of these equations will be linearly dependant on the others. So we have a total of 4 linearly independent equations and hence we can expect a unique solution in general. Returning to the equations for the composite variables $m_1x_1, \ldots, m_2y_2$, these can now be solved for the masses m_1 and m_2. □

The above can be given a geometric interpretation in $\mathbb{P}^9$. Restricting to a plane $z = z_i w$ in $\mathbb{P}^3$ corresponds to intersecting the Veronese variety with a 5-plane. The result will be a 2-dimensional Veronese variety usual called the Veronese surface. The two planes $i = 1, 2$, give two Veronese surfaces meeting in a conic—the image of the line $z = w = 0$. Varying the masses m_1 and m_2 for two points on these surfaces gives a line of inertia matrices. The space X, of all these lines the join of the two Veronese surfaces. Finally, the original inertia matrix is a point in $\mathbb{P}^9$ and we can take the cone over X with the original inertia matrix as vertex. This six dimensional variety corresponds to the space of all possible inertias we can produce by adding two point masses on planes $z = z_1w$ and $z = z_2w$ to the original body. The theorem asserts that this variety meets the 3-plane of inertias symmetrical with respect to an axis parallel to the z-axis in a single point.

6. Conclusions

The above represents the first steps in trying to introduce modern geometrical methods into the classical subject of equimomental systems.

It is likely that the ideas outlined here can also be applied to the problem of synthesising systems of springs to produce a given stiffness matrix.

REFERENCES

Asada H. and Youcef-Toumi, K. C. 1986 *Direct Drive Robots: Theory and Practice.* MIT.

Chaudhary, H. and Saha, S.K. 2009 *Dynamics and Balancing of Multibody Systems* vol. 37, Lecture Notes in Applied and Computational Mechanics. Springer.

Harris, J. 1992 *Algebraic geometry a first course.* Springer.

Huang, N. C. 1993 Equimomental systems of rigidly connected equal particles. *J. Guidance* vol. 16, part 6, pp. 1194–1196.

Laub, A.J. 2004 *Matrix Analysis for Scientists and Engineers.* SIAM.

Routh, E.J. 1950 *Dynamics of Rigid Bodies, Elementary Part.* Dover.

Selig, J. M. 2005 *Geometric Fundamentals of Robotics(2nd edition).* Springer.

Selig, J. M. and Martin D. 2014 On the line geometry of rigid-body inertia. *Acta Mechanica* vol. 225, part 11, pp. 3073-3101.

Sommerville, D. M. Y. 1930 Equimomental tetrads of a rigid body. *Mathematical Notes* vol. 26, pp. x–xi.

Stokes, A. and Brockett, R. 1996 Dynamics of kinematic chains. *International Journal of Robotics Research* vol. 15, part 4, pp. 393–405.