# Signatures of Systems with Non-exchangeable Lifetimes: Some Implications in the Analysis of Financial Risk

**Roy Cerqueti and Fabio Spizzichino**



**Abstract** We review the basic aspect of the concept of signature for a coherent system. The cases of exchangeability and non-exchangeability are compared in view of possible applications to the analysis of financial risk. The case of a special class of basket option is finally analyzed.



**Keywords** ■■■

## 1 Introduction

The concept of *signature*, introduced by [12], is a simple and useful tool for the analysis of a reliability system. Since its first inception, the relevance of this concept when dealing with "coherent systems" (see [2]) became evident. For a wide review of this topic we address the reader to the references cited in the bibliography and, in particular, to [3, 5, 6, 13].

One basic problem in system reliability lies in the analysis of the relationship between the reliability of a system and that of each of its single components. The concept of signature produces, in a sense, a change of perspective and focuses on the (random) number of components' failures that lead the system to its own failure.

Initially, signature has been employed under the condition of components with independent and identically distributed lifetimes. Such a concept, in fact, is specially relevant in that case, where a large part of the casualty in the system's lifetime is induced by the casuality in the temporal order in which the different components fail. Systems with i.i.d. components, on the other hand, do not always fit with

R. Cerqueti (✉)
Department of Economics and Law, University of Macerata,
Via Crescimbeni 20, 62100 Macerata, Italy
e-mail: roy.cerqueti@unimc.it

F. Spizzichino
Department of Mathematics, Sapienza University of Rome,
Rome, Italy
e-mail: fabio.spizzichino@uniroma1.it



C. Kitsos et al. (eds.), *Theory and Practice of Risk Assessment*,
Springer Proceedings in Mathematics & Statistics 136,
DOI 10.1007/978-3-319-18029-8_27

real-world applications. It has then been noticed that the definition of signature can be extended, in a completely natural way, to the case of exchangeable components (see [9, 10]). This extension is particularly important, because it allows us to consider components' lifetimes that are conditionally i.i.d. rather than just i.i.d.

As a further generalization, more recent studies dealt with the concept of signature even in cases of non-exchangeability (see e.g. [8, 11, 15]). The case of non-exchangeability leads to two different concepts of signature: the first one is only related to the structure of the system, while the other one is related to both the structure of the system and to the joint distribution of the components lifetimes. The former is concerned with the symmetry properties of the system [15], while the latter can play a role in the computation and approximation of the system reliability in particular (see e.g. [8, 11, 15]).

To the best of our knowledge, the concept of signature has been employed so far exclusively in the field of reliability systems. The case of non-exchangeable components provides however a realistic representation of a wider family of systems and networks appearing in the applied sciences. The possibility of extending this concept to non-exchangeable cases, open then the path to applications to other fields. In particular we think that signature can play a useful role in the field of Economics and financial risk even if this path remains unexplored.

As a preliminary task in the direction of filling this gap, it is important to understand the differences, as far as properties and meaning of signatures are concerned, between the two cases of exchangeability and non-exchangeability.

In this note we deal with some aspects of this issue and point out some relevant implications. Some of such implications will be also demonstrated by considerations of financial character.

More precisely, the remaining part of the paper is organized as follows. In Sect. 2 we briefly recall the concepts of signatures and present preliminaries and notation. In Sect. 3 we discuss the main differences between the cases of exchangeable and non-exchangeable lifetimes. A discussion about some related aspects from the point of view of financial applications, will be presented in the Sect. 4, with a specific focus on basket options.

## 2 Preliminaries, Notation, and Definitions of Signatures

We consider a *reliability system* $S$ formed by $n$ *components* $C_1, \ldots, C_n$. Given $j = 1, \ldots, n$ and a time $t > 0$, the *status of the* $j$*th component* at time $t$ is a binary variable $Y_j(t)$ defined by

$$Y_j(t) = \begin{cases} 1 & \text{if } \mathrm{C_j} \text{ is working at time t} \\ 0 & \text{if } \mathrm{C_j} \text{ is down at time t.} \end{cases}$$

Each component is assumed to be working at time $t = 0$, and hence $Y_j(0) = 1$, for each $j = 1, \ldots, n$. The *status of the system* can analogously be defined by letting

$$Y_S(t) = \begin{cases} 1 & \text{if S is working at time t} \\ 0 & \text{if S is down at time t,} \end{cases}$$

Fix $t \geq 0$. One assumes that $Y_S(t)$ is exclusively determined by $Y_1(t), \ldots, Y_n(t)$ and defines the *structure function of the system* as the function $\varphi_S : \{0, 1\}^n \to \{0, 1\}$ such that:

$$Y_S(t) = \varphi_S \left( Y_1 \left( t \right), \ldots, Y_n \left( t \right) \right).$$

$\varphi_S$ is usually assumed to be *coherent*, i.e. the following conditions are satisfied:

- $\varphi_S(0, \ldots, 0) = 0$, $\varphi_S(1, \ldots, 1) = 1$;
- $\varphi_S$ is non-decreasing with respect to its components;
- each component of $S$ is *relevant*

Now, denote by $\mathscr{G}$ the set of the *path vectors* of the system, i.e.

$$\mathscr{G} = \{\mathbf{y} \in \{0, 1\}^n | \varphi_S (\mathbf{y}) = 1\}.$$

Trivially $\mathbf{y} = (1, \ldots, 1)$ is a path vector and thus $Y_S(0) = 1$.

The *lifetime* of $S$ and that of the $j$th component are respectively given by $X_S$ and $X_j$, where

$$X_S = \inf\{t \geq 0 \,|\, Y_S(t) = 0\} = \inf\{t \geq 0 \,|\, (Y_1(t), \ldots, Y_n(t)) \notin \mathscr{G}\},$$

and

$$X_j = \inf\{t \geq 0 \,|\, Y_j(t) = 0\}.$$

Furthermore, $R_S(t)$ denotes the *reliability function* of the system at time $t$, namely:

$$R_S(t) \equiv P\{X_S > t\}, \qquad \forall\, t \geq 0. \tag{1}$$

The term $R_S(t)$ depends both on the structure function $\varphi_S$ and on the joint distributions of the components' lifetimes. As we are going to discuss in the following, the concept of signature provides an insight about the structure of such dependence.

We first recall the formal definitions of *structure signature* and *probability signature*. For this purpose we need the following further assumptions and notation.

First of all, it is convenient to imagine that each component continues to work until its own failure, even if the system has already failed, so that all the lifetimes $X_1, \ldots, X_n$ are well defined and can be eventually observed. We assume furthermore that the joint distribution of the elements of the vector $\mathbf{X} = (X_1, \ldots, X_n)$ is such that

$$P\{X_1 \neq \cdots \neq X_n\} = 1. \tag{2}$$



334259_1_En_27_Chapter ✓ TYPESET ☐ DISK ☐ LE ✓ CP Disp.:16/4/2015 Pages: 15 Layout: T1-Standard

By considering the order statistics $X_{(1)}, \ldots, X_{(n)}$ of the vector $\mathbf{X}$ we thus have:

$$P\{X_S = X_{(k)}\}, \tag{3}$$

for one and only one $k = 1, \ldots, n$.

Before continuing, let us remark that the failures of the subsequent components give rise to the progressive observation of a permutation of $\{1, \ldots, n\}$. All the $n!$ possible permutations can be observed and each permutation describes a possible temporal order in which the different components fail.

Consider the events $E_k$, defined by

$$E_k \equiv \{X_S = X_{(k)}\}, \qquad k = 1, \ldots, n. \tag{4}$$

$E_1, \ldots, E_n$ form then a partition of the sample space, i.e. one and only one of them will be observed.

Denote by $\mathscr{P}$ the set of all the permutations of $\{1, \ldots, n\}$ and consider the random vector $(J_1, \ldots, J_n)$ defined by:

$$J_k = i \text{ when } X_{(k)} = X_i, \qquad \forall k = 1, \ldots, n, \tag{5}$$

i.e. $J_k$ indicates the identity of the component which fails in correspondence of the $k$th observed failure. We also set

$$A_k \equiv \{(j_1, \ldots, j_n) \in \mathscr{P} \mid J_1 = j_1, \ldots, J_n = j_n \Rightarrow X_S = X_{(k)}\}.$$

While the events $E_1, \ldots, E_n$ form a partition of the sample space, the sets $A_1, \ldots, A_n$ form a partition of the set $\mathscr{P}$ and we have

$$\sum_{k=1}^{n} |A_k| = n!.$$

As to the logical relation between these two partitions, we can write

$$\{(J_1, \ldots, J_n) \in A_k\} = \{X_S = X_{(k)}\} \equiv E_k.$$

One basic remark is that $(A_1, \ldots, A_n)$ is determined by the structure function $\varphi_S$.

We can now recall the definitions of two different notions of signatures

**Definition 1** • The *structure signature* of $S$ is $\mathbf{p} \equiv (p_1, \ldots, p_n)$, where

$$p_k = \frac{|A_k|}{n!}, \qquad k = 1, \ldots, n.$$

- The *probability signature* of $S$ is $\hat{\mathbf{p}} \equiv (\hat{p}_1, \ldots, \hat{p}_n)$, where

$$\hat{p}_k = P(E_k), \qquad k = 1, \ldots, n.$$

See [8, 10–12, 15]. Concerning the events $\{E_k\}_{k=1,\ldots,n}$ we can write, by recalling (4),

$$X_S = \sum_{k=1}^{n} X_{(k)} \mathbf{1}_{E_k}, \tag{6}$$

and, by applying the law of total probabilities and by (1), (4) and (6), we can conclude:

$$R_S(t) \equiv \sum_{k=1}^{n} P(E_k) \cdot P\{X_{(k)} > t | E_k\}. \tag{7}$$

In view of the definitions above, the decomposition (7) can be rewritten in terms of the probability signature:

$$R_S(t) \equiv \sum_{k=1}^{n} \hat{p}_k \cdot P\{X_{(k)} > t | E_k\}. \tag{8}$$

## 3 Two Different Scenarios and Different Roles of Signatures

Concerning the two concepts of probability signature and of structure signature we observe different properties depending on the type of joint distribution that is assessed for the lifetimes of the components. As mentioned above, the scenario obtained under the condition of exchangeability is fairly special and it is rather different from the one that emerges in the non-exchangeable cases. Even the relations existing between the two concepts and their roles in applied problems are generally different in the two cases. These differences will be briefly outlined in this section, where the cases of exchangeability and non-exchangeability will be treated separately. The condition (2) is however assumed in any case, since it is necessary for the definitions of signatures to be meaningful. More arguments on this topic can be found in the cited references; some potentially useful examples are discussed in [16].

Before starting our discussion here, it is useful to pay attention to a couple of simple remarks. First, we notice that, from a purely mathematical viewpoint, both the vectors $\mathbf{p}$ and $\hat{\mathbf{p}}$ can be seen as probability distributions over the space $\{1, \ldots, n\}$. The probability signature $\hat{\mathbf{p}}$, in particular, can be seen as the probability distribution of the random variable $M$, defined as follows: $M$ is the number of the observed component failures up to the failure of the system. A very special class of coherent systems is relevant in the reliability field and in a signature-based analysis, in particular. This is the class of systems of the type $k : n$ (for $k = 1, \ldots, n$). A system $k : n$ is one which is able to work as long as at least $k$ of its components are working, namely it fails at the instant of the $(n-k+1)$th components' failure. In particular, a *parallel system* is a system $1:n$ and a *series system* is a $n:n$ system. In the case of a $k:n$ system we have $P(M=n-k+1)=1$ and both $\mathbf{p}$, $\hat{\mathbf{p}}$ are degenerate probability distributions, with $p_{n-k+1}=\widehat{p}_{n-k+1}=1$. Notice also that, in these cases, the structure of the system is perfectly symmetric. In other words all the components in the system contribute in a same way in maintaining the system in its working state.

## 3.1 The Exchangeable Case

We first consider the case where the components' lifetimes $X_1,\ldots,X_n$ are exchangeable random variables. Namely the joint distribution of $\mathbf{X}$ is invariant with respect to permutations of the variables. As an immediate consequence of this assumption, the random permutation $(J_1,\ldots,J_n)$, defined in (5), is distributed uniformly over $\mathscr{P}$, i.e.:

$$P\{(J_1,\ldots,J_n)\in B\}=\frac{|B|}{n!},\qquad \forall\, B\subseteq\mathscr{P}.$$

This entails the following simple result (see e.g. the discussion in [15]).

**Proposition 1** *1. For $k=1,\ldots,n$, one has*

$$\hat{p}_k=p_k;$$

*2. the events $\left(X_{(k)}>t\right)$ and $E_k$ are independent;*
*3. the reliability function of the system is:*

$$R_S(t)=\sum_{k=1}^{n}p_k P\{X_{(k)}>t\}. \tag{9}$$

We notice that item 3. is an immediate consequence of 1. and 2. and of the total probability formula (8). Moreover, items 1. and 2 are immediate consequences of the assumption that all the permutations $(j_1,\ldots,j_n)$ are equally probable as values for $(J_1,\ldots,J_n)$. We recall that each permutation describes a different temporal order in which the different components fail.

The statements in Proposition 1 are relevant from an applied point of view. From 1. we see that, in the exchangeable case, structure signature and reliability signature collapse into one and the same concept. Thus the probability distribution of the random variable $M$ only depends on the structure of the system and it is not influenced by the joint probability law of the lifetimes $X_1,\ldots,X_n$. This lack of interaction is confirmed by item 2.

Let us now examine item 3. in details. It is clear that $R_S(t)$ generally depends on the pair $(\varphi_S,F_{\mathbf{X}})$ where $\varphi_S$ is the structure of the system and $F_{\mathbf{X}}$ denotes the joint probability distribution function of the lifetimes $X_1, \ldots, X_n$. Such dependence may turn out to be rather complex, in some cases. The special form (9) of the formula of total probabilities (8) has then the following interpretation: when $X_1, \ldots, X_n$ are exchangeable (9) shows that $R_S(t)$ depends on $\varphi_S$ only through the system signature $\mathbf{p}$ (which is only a function of $\varphi_S$ and is it is not influenced by $F_{\mathbf{X}}$). On the other hand, $R_S(t)$ is influenced by $F_{\mathbf{X}}$ only through the vector of the marginal distributions of the order statistics $X_{(1)}, \ldots, X_{(n)}$.

These facts entail the following implications:

1. Consider two coherent systems $S'$ and $S''$ formed with different sets of components $C'_1, \ldots, C'_n$ and $C''_1, \ldots, C''_n$, respectively, and such that they share the same structure functions, i.e. $\varphi'_S \equiv \varphi''_S$. Then, as long as the vectors of the components' lifetimes are exchangeable, $S'$ and $S''$ share the same (probability and structure) signature, even if the joint distributions are different.
2. Think of a coherent system $S$, all the components of which play similar roles as to the system's capability to work. In such a case, we are allowed to interchange the respective positions of any two components in the system. This situation is met, for instance, in a network where all the components are just transmission nodes, possibly with different capacities but similar in nature. For such a system $S$, consider a permutation $\pi \in \mathscr{P}$ and denote by $\mathbf{S}_\pi$ the system obtained by permuting the components through $\pi$. Then the reliability functions $R_S(t)$ and $R_{S_\pi}(t)$ coincide, for any $t$.

## 3.2 The Non-exchangeable Case

In this subsection we consider the case when $X_1, \ldots, X_n$ are not exchangeable, so that we cannot rely anymore on Proposition 1. As a first consequence, the structure signature and the reliability signature do not necessarily coincide. We can still consider the structure signature $\mathbf{p}$ which, by definition is a combinatorial invariant, only determined by the structure $\varphi_S$. But this vector does not carry complete information about the probabilities $\widehat{p}_1, \ldots, \widehat{p}_n$. Actually, the vector $\hat{\mathbf{p}}$ is influenced also by the choice of the joint distribution function $F_{\mathbf{X}}$. Moreover, the formula (8) cannot be reduced to (9). Generally both the vectors $\hat{\mathbf{p}}$ and $\left(P\{X_{(1)} > t|E_1\}, \ldots, P\{X_{(n)} > t|E_n\}\right)$, whose scalar products produce $R_S(t)$, depend on both the data $\varphi_S$, $F_{\mathbf{X}}$.

It is now interesting to briefly point out the different roles of $\hat{\mathbf{p}}$ and $\mathbf{p}$ in reliability problems.

$\hat{\mathbf{p}}$ can be applied in different ways. It can be used in particular for defining the *projected system*, which provides in a sense the best approximation of the original system [8, 11]. Furthermore it could be used for extending to the non-exchangeable case comparisons, between two systems, that have been developed for i.i.d components and that are based on the structural signature. See also below.

For the purpose of analyzing the possible role of $\mathbf{p}$, it is again convenient to consider a coherent system $S$ whose components have similar roles, so that interchanging the respective positions of any two components makes sense. For these cases we would like to investigate what happens if we permute, according to some permutation $\pi \in \mathscr{P}$, the positions of the components.

Fix a permutation $\pi$ and recall that $S_\pi$ denotes the new system, obtained by applying $\pi$ on the components of $S$. The structure function of $S_\pi$ is just given by

$$\varphi_\pi(\mathbf{y}) = \varphi(\mathbf{y}_\pi)\,. \tag{10}$$

Since the reliability function depends on the probability signature and the latter depends on the joint distribution function of the lifetimes then, generally, $R_{S_\pi}(t) \neq R_S(t)$, for $t > 0$. We denote by $R^*(t)$ the *symmetrized* reliability function defined as follows:

$$R^*(t) = \frac{1}{n!}\sum_{\pi\in\mathscr{P}} R_\pi(t)\,. \tag{11}$$

Notice that we implicitly identified $R_{(1,\ldots,n)}$ with $R_S$, where $\{1,\ldots,n\}$ is the identical permutation.

It is also useful to adopt the notation $R_S^{(F)}(t)$, in order to stress the dependence of the reliability function on the joint law $F$ of the components lifetimes $X_1,\ldots,X_n$. Furthermore we denote by $F_\pi$ the joint law of the permuted vector $\mathbf{X}_\pi$. One can see that

$$R_\pi^{(F)}(t) = R_{\{1,\ldots,n\}}^{(F_\pi)}(t).$$

Denote now by $\Pi$ a random permutation of $\{1,\ldots,n\}$, uniformly distributed over $\mathscr{P}$, and set

$$\left(X_1^*,\ldots,X_n^*\right) = \left(X_{\Pi_1},\ldots,X_{\Pi_n}\right).$$

Finally we denote by $F^*$ the joint distribution function of $\left(X_1^*,\ldots,X_n^*\right)$ and by $R_S^{(\mathbf{F}^*)}(t)$ the reliability function of the system $S$ when the lifetimes of its components are $\left(X_1^*,\ldots,X_n^*\right)$. The random vector $\left(X_1^*,\ldots,X_n^*\right)$ is exchangeable and it is such that the vectors of the order statistics $\left(X_{(1)}^*,\ldots,X_{(n)}^*\right)$ and $\left(X_{(1)},\ldots,X_{(n)}\right)$ share the same joint law. All these properties and positions lead us to the following result.

**Proposition 2** *All the systems $S_\pi$, for $\pi \in \mathscr{P}$, share the same structure signature* $\mathbf{p}$. *Furthermore*

$$R_S^{(\mathbf{F}^*)}(t) = \sum_{k=1}^{n} p_k P\{X_{(k)} > t\}, \tag{12}$$

$$R^*(t) = R_S^{(\mathbf{F}^*)}(t)\,. \tag{13}$$

See [15] for details. Thus we see that $R^*(t)$ can be interpreted as the reliability function of a fictitious system (the *average system*), having the same structure of $S$ and same components of $S$; but such that the components are distributed at random



334259_1_En_27_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:16/4/2015 Pages: 15 Layout: T1-Standard

among the different positions. As shown by (12) $R^*(t)$ is, typically, more easily computed than $R(t)$. Even if its meaning is fictitious it can still be of interest. Consider in this respect the function

$$\mu(t) = |R_S^{(\mathbf{X})}(t) - R^*(t)|. \quad (14)$$

We expect that $R_S^{(\mathbf{X})}(t) - R^*(t) \geq 0$ when the system is correctly designed. The function $\mu(t)$ expresses a sort of distance between the reliability function and the *symmetrized* reliability function $R^*(t)$ for the system $S$. It is related to the amount of asymmetry of the system $S$: the larger the symmetry level of the structure function $\varphi_S$, the smaller the difference in the left-hand side of (14). On the other hand, it can be argued that the more the structure signature is a concentrated probability distribution, the smaller is the asymmetry of the system. Recall in this respect that, as we had noticed above, degenerate signatures, in particular, correspond to the completely symmetric structures of the type $k : n$. We then see that the structure signature has a double role: it allows us to compute $R^*(t)$ by means of (12) and provides us with some information about the error that arises, in the computation of the reliability function, when we approximate $R(t)$ by $R^*(t)$, namely when we replace the "true" distribution of $X_1, \ldots, X_n$ with the exchangeable distribution which gives rise to the same joint distribution for the order statistics.

Let $S'$, $S''$ be two systems with the same number of components and let $\mathbf{p}'$, $\mathbf{p}''$ be their structural signatures respectively. As already mentioned $\mathbf{p}'$, $\mathbf{p}''$ also permit one to compare $S'$, $S''$ in the following sense: different types of stochastic orderings between the probability distributions $\mathbf{p}'$, $\mathbf{p}''$ imply corresponding stochastic orderings between the reliability functions of $S'$, $S''$, when a vector of the same i.i.d. components is installed in the two systems (see [6]). This can be a good way to compare the two systems, even for cases when the components are not exchangeable. Furthermore one can conjecture that results similar to those in [6] could be extended to non-exchangeable case, in terms of $\widehat{\mathbf{p}}'$, $\widehat{\mathbf{p}}''$.

## 4 A Special Class of Basket Options and Implications of Non-exchangeability

In this section we focus attention on financial applications and, more precisely, on the risk associated to the so-called *basket options*. On one hand we point out that the topic of signature can be of some interest also in this field. On the other hand we further discuss, just from an economic viewpoint, the implications related with the difference between exchangeability and non-exchangeability, as far as signature is concerned.

Basket options constitute one of the most popular and traded structured products, and belong to the wide family of *exotic options* (see [17]). The success of this financial product lies in low prices, in the management of the risk profile through



334259_1_En_27_Chapter ✓ TYPESET DISK LE ✓ CP Disp.:16/4/2015 Pages: 15 Layout: T1-Standard

an appropriate selection of correlated assets in the basket and in the reduction of the transaction costs. The payoff of this product is linked to the performance of a collection (basket) of assets. On such a basis, the option may be of various typologies in nature. We will consider here a particular model of basket options, where the basket is composed of a set of $n$ assets, formed with a subset of $r$ “important” assets and a set of $s$ “standard” assets, $n = r + s$. For all the assets, irrespectively of whether they are important or not, a lower barrier is considered which should not be crossed until the maturity time of the option (see e.g. [1, 4, 7]).

We can think of an important asset as one for which a very big amount of stocks is traded on the market. We can then expect that its volatility is smaller than that of the assets with less stocks and this may reflect in a lower riskiness.

Let $T > 0$ be the expiration time (or time to maturity) for the option and $\alpha > 0$ be the common barrier for all the assets in the basket. Furthermore, for $t \geq 0$ and $j = 1, \ldots, n$, let $\Lambda_j(t)$ be the stochastic process describing the evolution of the return of the $j$th asset. We consider then the $n$-dimensional vector of (random) failure times $\mathbf{X} = (X_1, \ldots, X_n)$ such that:

$$X_j = \inf\{t > 0 \mid \Lambda_j(t) \leq \alpha\}. \tag{15}$$

$X_j$ will be then interpreted hereafter as the lifetime of the $j$th asset and it can be also convenient to set

$$Y_j(t) = \begin{cases} 1 & \text{if } X_j > t, \\ 0 & \text{otherwise.} \end{cases}.$$

A basket option will be viewed as a coherent system $S$ whose $n$ components $C_1, \ldots, C_n$ are the assets in the basket. Once the financial structure of the option has been fixed, one defines the failure time of the option a random variable $X_S$, suitably defined as a function of $X_1, \ldots, X_n$.

At the expiration time $T$ the holder of the option obtains a return $Ret_T > 0$, under the condition

$$X_S > T.$$

For $t \geq 0$, the reliability function of the option at time $t$ is

$$R_S(t) \equiv P\{X_S > t\}.$$

Generally, the price of a financial product is clearly related with its risk level. For our basket option, an appropriate measure of riskiness is the value $R_S(T)$, which then plays a relevant financial role.

In order to exactly define the very nature of the options that we consider or, in other words, to describe the structure function of the system, we in particular focus attention on financial models defined in terms of a nonincreasing function

$$\rho : \{1, \ldots, n\} \to \{0, 1, \ldots, r + 1\},$$



334259_1_En_27_Chapter ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:16/4/2015 Pages: 15 Layout: T1-Standard

satisfied the condition with a meaning described as follows: the option has a fatal default at the first time in which the failures of $k$ assets are observed, with $k$ such that at least $\rho(k)$ failures are due to the more important assets. It is natural to assume that the function $\rho(k)$ is nonincreasing. A few more precise details about its definition are however in order.

The condition $\rho(k) = 0$ obviously means that the failure of $k$ standard assets is enough to determine the default of the option. The position $\rho(k) = r + 1$ means that $k$ is so small that the failure of $k$ assets cannot produce the option's default, even in the case when all the failed assets are important ones. The minimum number of failures able to determine the default is the minimum value of $k$ that satisfies the condition $\rho(k) \leq k$. The maximum possible number of failures that can be conceptually observed up to the default coincides with the minimum value of $k$ such that $\rho(k) = 0$.

Let us now proceed to formally define the option's default time $X_S$.

Set

$$N_k \equiv \sum_{j=1}^{r} \left(1 - Y_j \left(X_{(k)}\right)\right).$$

$N_k$ then denotes the number of assets that have already failed at the moment of the $k$th overall failure. We let

$$X_S = X_{(k)}$$

if and only if

$$N_k \geq \rho(k), \;\; N_h < \rho(h),$$

for $h = 1, \ldots, k - 1$.

In other words, the family of the path vectors of the systems is defined by

$$\left\{ \mathbf{y} \in \{0, 1\}^n \mid r - \sum_{j=1}^{r} y_j < \rho \left( n - \sum_{j=1}^{n} y_j \right) \right\}. \tag{16}$$

We notice that such a system manifests the following structure of partial symmetry: all the important assets share a common role and also all the standard assets share a common role of their own. In a sense this structure could be seen as a natural generalization of the famous $k$-out-of-$n$ models. To designate our models, we may use the term $(n - \rho(k))$*-out-of-n systems*.

*Remark 1* In the field of basket options, a further generalization could be sometimes more realistic: one may admit that the above numbers $\rho(k)$ are replaced by numbers $\rho(k; J)$ also depending on the subsets $J \subset \{1, \ldots, s\}$ of standard assets that failed up to the time $X_{(k)}$. The assumption that $\rho(k)$ is a non-decreasing function of $k$, should be replaced by a new condition involving also the monotonicity with respect to $J$. Models of this type are also related to the concept of system with weighted components, analyzed in [14].

To the best of our knowledge, coherent systems of the type $(n-\rho(k))$-out-of-$n$ have not been considered so far from the point of view of a signature analysis. The following result shows the form of their structure signature $\mathbf{p}=(p_1,\ldots,p_n)$. Denote by $I_\rho$ the set $\{k\in\{1,\ldots,n\}\,|\,0<\rho(k)\le k\}$.

**Proposition 3** *(a) Let $k \in I_\rho$. Then:*

$$p_k=\sum_{j=0}^{\rho(k-1)-1}\frac{\binom{r}{j}\binom{n-r}{k-j-1}}{\binom{n}{k-1}}-\sum_{j=0}^{\rho(k)-1}\frac{\binom{r}{j}\binom{n-r}{k-j}}{\binom{n}{k}}; \qquad (17)$$

*(b) $p_k=0$ if $\rho(k)=r+1$;*
*(c) $p_k=0$ if $\rho(k)=\rho(k-1)=0$;*
*(d) Let $k$ be such that $\rho(k)=0$, $\rho(k-1)>0$. Then*

$$p_k=1-\sum_{h\neq k}p_h.$$

*Proof* (a) First, we recall that the structure signature of a system coincides with the probability signature, where the latter is computed under the assumptions that the components are i.i.d. Thus we need to compute the probabilities

$$P\left(X_S=X_{(k)}\right),\qquad k=1,\ldots,n,$$

under the assumption that the assets' lifetimes $X_1,\ldots,X_n$ are i.i.d.
For $k\in I_\rho$, we consider the quantity $\overline{P}_k:=\sum_{h=k+1}^{n}p_h$, so that

$$\overline{P}_k=P\left(X_S>X_{(k)}\right)=P\left(N_k<\rho(k)\right)=\sum_{j=0}^{\rho(k)-1}P\left(N_k=j\right).$$

Then

$$p_k=\overline{P}_{k-1}-\overline{P}_k=\sum_{j=0}^{\rho(k-1)-1}P\left(N_{k-1}=j\right)-\sum_{j=0}^{\rho(k)-1}P\left(N_k=j\right).$$

In view of the assumption that the assets' lifetimes $X_1,\ldots,X_n$ are i.i.d., the terms $P\left(N_k=r\right)$ are given by hypergeometric probabilities. More precisely:

$$P\left(N_k=j\right)=\frac{\binom{r}{j}\binom{n-r}{k-j}}{\binom{n}{k}}.$$

(b) The condition $\rho(k) = r + 1$ means that the observation of $k$ failures cannot cause the default of the option. Thus $p_k = 0$.
(c) If $\rho(k) = 0$ then $k$ failures cause the default of the option, if the latter had not defaulted before. Thus the probability of a default at $X_{(k)}$ is null when $\rho(k-1) = 0$.
(d) It trivially follows from (a), (b) and (c), since $\sum_{k=1}^{n} p_k = 1$.

As already discussed in Sect. 2, the signature analysis of a system is strongly influenced by the conditions of exchangeability or non-exchangeability among the components.

In the present context, exchangeability of $X_1, \ldots, X_n$ is reflected by a symmetry condition among the behavior of the assets' returns $\Lambda_1, \ldots, \Lambda_n$ and the following statement can in particular be made: at any fixed time $t$, the probability that $h < n$ returns are above the threshold $\alpha$, while the remaining $n - h$ returns are below $\alpha$, is independent on the specific selection of the $h$ assets.

We are in the non-exchangeability case when such a statement is no longer true. In this respect, non-exchangeability can be viewed as a condition of "heterogeneity" among the assets of the basket. Specifically, in analyzing the joint behavior of the assets at the expiration date $T$, the identity of any single asset matters. This is actually a typical circumstance in the above setting. Exchangeability is then only an extreme and idealized condition, for us.

Let us briefly mention some relevant implications of non-exchangeability on the signature analysis.

As a first remark, we can say that the special structure $(n - \rho(k))$-out-of-$n$ is just appropriate for the financial model of heterogeneity, where the assets can be of only two "types".

We can moreover recall that the probability signature is different from the structure signature detailed in (17). When the important assets are more reliable than the other ones, the probability signature is stochastically larger than the structure signature. This circumstance would guarantee that the "projected system" is less risky than the "average system", where the "projected system" provides a better approximation of the reliability of the system (of the option, in our case) than the "average system" [11].

A further remark concerns the effect of some possible piece of new information about the market. Suppose that short after time 0, an event $A$ is observed that modifies the evaluation of the future performance of the assets (such as e.g. the failure of an important asset, outside the basket). This has a double effect on the terms in the r.h.s. of Eq. (8). Not only the factors $P\{X_{(k)} > t | E_k\}$ change into $P\{X_{(k)} > t | E_k \cap A\}$, but also the weights $\hat{p}_k$ are influenced by the replacement of joint distribution (prior to $A$) with a different one (posterior to $A$), when at least one of the two is not exchangeable. This circumstance may have the following relevant consequence. On the basis of a same set of assets, consider two different options $O_1$ and $O_2$, characterized by different and non-comparable functions $\rho_1(k)$ and $\rho_2(k)$. Compare then $O_1$ and $O_2$ in terms of their levels of riskiness and then in terms of their price: it can happen that the ordering between $O_1$ and $O_2$ posterior to $A$ is the opposite of the ordering if the comparison had been made prior to $A$. In view of the validity of the formula (9), this situation cannot manifest when the prior and posterior joint distributions are both exchangeable.

The condition of non-exchangeability is even more intrinsic to the nature of the option, when we consider the models mentioned in Remark 1. In such models a character of heterogeneity is present and it makes sense to compare two different options obtained by different arrangements in the system of a same set of assets. The problem then arises of determining the most efficient permutation. It is useful to recall in this respect that the structure signature and probability signature are of help in such an analysis. The fact that probability signature can be influenced by arrival of new information can be an interesting issue for further research.

## References

1. Alexander, C.: Market Models. Wiley, West Sussex (2001)
2. Barlow, R.E., Proschan, F.: Statistical Theory of Reliability and Life Testing, To Begin With, Silver Spring, Maryland (1981)
3. Boland, P.J., Samaniego, F.J.: The signature of a coherent system and its applications in reliability. In: Mazzucchi, T., Singpurwalla, N., Soyer, R. (eds.) Mathematical Reliability: An Expository Perspective, pp. 3–30. Kluwer Academic Publishers, Boston (2004)
4. Cerqueti, R., Rotundo, G.: Options with underlying asset driven by a fractional Brownian motion: crossing barriers estimates. New Math. Nat. Comput. **6**(1), 109–118 (2010)
5. Gertsbakh, I.B., Shpungin, Y.: Models of Network Reliability. CRC Press, Boca Raton (2010)
6. Kochar, S., Mukerjee, H., Samaniego, F.J.: The signature of a coherent system and its application to comparisons among systems. Nav. Res. Logist. **46**(5), 507–523 (1999)
7. Kou, S.G., Wang, H.: Option pricing under a double exponential jump diffusion model. Manag. Sci. **50**(9), 1178–1192 (2004)
8. Marichal, J.-L., Mathonet, P.: Extensions of system signatures to dependent lifetimes: explicit expressions and interpretations. J. Multivar. Anal. **102**(5), 931–936 (2011)
9. Navarro, J., Rychlik, T.: Reliability and expectation bounds for coherent systems with exchangeable components. J. Multivar. Anal. **98**, 102–113 (2007)
10. Navarro, J., Balakrishnan, N., Bhattacharya, D., Samaniego, F.: On the application and extension of system signatures in engineering reliability. Nav. Res. Logist. **55**, 313–327 (2008)
11. Navarro, J., Spizzichino, F., Balakrishnan, N.: Applications of average and projected systems to the study of coherent systems. J. Multivar. Anal. **101**(6), 1471–1482 (2010)
12. Samaniego, F.J.: On closure of the IFR class under formation of coherent systems. IEEE Trans. Reliab. **R34**, 60–72 (1985)
13. Samaniego, F.J.: System Signatures and Their Applications in Engineering Reliability. International Series in Operations Research and Management Science, vol. 110. Springer, New York (2007)
14. Samaniego, F.J., Shaked, M.: Systems with weighted components. Stat. Probab. Lett. **78**(6), 815–823 (2008)
15. Spizzichino, F.: The role of signature and symmetrization for systems with non-exchangeable components. In: Advances in Mathematical Modeling for Reliability, pp. 138–148. IOS, Amsterdam (2008)
16. Spizzichino, F., Navarro, J.: Signatures and symmetry properties of coherent systems. In: Recent Advances in System Reliability. Springer Series in Reliability Engineering, pp. 33–48. Springer, New York (2012)
17. Zhang, P.G.: Exotic Options. World Scientific Publishing Co., Singapore (1997)



334259_1_En_27_Chapter ✓ TYPESET DISK LE ✓ CP Disp.:16/4/2015 Pages: 15 Layout: T1-Standard