Non exchangeable copulas and multivariate total positivity

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Abstract

Multivariate total positivity of order 2 (MTP2) is a dependence property with a number of applications in statistics and mathematics. Given the theoretical and practical relevance of MTP2, it is important to investigate the conditions under which random vectors have this property. In this paper we contribute to the development of the theory of stochastic dependence by employing the general concept of copula. In particular, we propose a new family of non-exchangeable Archimedean copulas which leads to MTP2. The focus on non-exchangeability allows us to overcome the limitations induced by symmetric dependence, typical of standard Archimedean copulas.

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1. Introduction

Total positivity is a concept of considerable interest in many fields of statistics and mathematics. In particular, multivariate total positivity of order 2 (MTP2, see [1]) has a number of applications in statistical decision procedures, multivariate analysis, simultaneous statistical inference, approximating probabilities, and reliability theory. The MTP2 property is known to be satisfied by a fairly limited number of multivariate distributions (see e.g. [1, 2]) and the concept of copula can be very useful in extending the family of known MTP2 random vectors. In this vein [3] derive necessary and sufficient conditions for the generator of an Archimedean copula to yield a random vector which is MTP2. It is important to note that Archimedean copulas (see e.g. [4, 5, 6, 7] for some recent results and extensive surveys), are among the most relevant examples of exchangeable copulas. Exchangeability is a very important property, satisfied by a qualified family of distributions. However, despite its mathematical relevance, exchangeability may represent a requirement too strong to be commonly fulfilled by a set of random variables. Therefore, it is worth paying attention to the concept of non-exchangeable generalization of Archimedean copulas (see [8, 9]).

In this paper, we move from [8, 9] and provide a new family of asymmetric copulas generated by a one-dimensional function which leads to MTP2 (see Theorem 3.3 below). In doing this, we extend [3] to the case of non-exchangeability for the considered vector of dependent random variables. Therefore, using Theorem 3.3 we can provide sufficient conditions for a vector of non-symmetrically dependent random variables to be MTP2: note also that the random variables are not required to possess a special joint distribution and can even have different marginals. The theoretical result leads to the identification of a new family of copulas associated to the MTP2 property (see Proposition 3.4 below).

The rest of the paper is organized as follows. Section 2 provides the necessary preliminaries, notation and statistical concepts. The main results are offered in Section 3. Some concluding remarks are given in Section 4.

2. Preliminaries and notation

For the sake of simplicity, we introduce the vectorial notation:

Notation 2.1. *Fix* $m = 1, 2, \ldots$. *The following notations are introduced:* $\boldsymbol{W} = (w_1, \ldots, w_m)$ *is a random vector;* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m)$ *are elements of* $\mathbb{R}^m$ *and* $\boldsymbol{u} = (u_1, \ldots, u_m) \in [0, 1]^m$.

We now recall the definition of the dependence concept we deal with.

Definition 2.2. *Let* f *be the joint density function of the* m*-variate random vector* $\boldsymbol{W}$. *The components of* $\boldsymbol{W}$ *are said to be MTP2 if and only if, for each* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in* $\mathbb{R}^m$, *it results:*

$$f(\boldsymbol{x}) \cdot f(\boldsymbol{y}) \leqslant f(\min\{\boldsymbol{x}, \boldsymbol{y}\}) \cdot f(\max\{\boldsymbol{x}, \boldsymbol{y}\})$$

where the min *and* max *operators are meant component-wise.*

Definition 2.2 formalizes a (not necessarily linear) dependence structure of positive type. When dealing with (linear) dependence among individual w's in $\boldsymbol{W}$, it is customary to consider a non-diagonal variance-covariance matrix $\boldsymbol{\Sigma} = (\sigma_{i,j})$ with $i, j = 1, \ldots, m$. Hence, it is natural to guess the existence of a relationship between the value (and the sign) of the covariances and the validity of MTP2. In this respect, it is worth recalling a standard result which states that if $\{w_1, \ldots, w_m\}$ are MTP2, then $\sigma_{i,j} \geqslant 0$, for each $i, j = 1, \ldots, m$ (see e.g. [1]). This fact implies that if there exists a couple (w_i, w_j), with $i \neq j$ and $i, j = 1, \ldots, m$, such that $\sigma_{i,j} < 0$, then $\{w_1, \ldots, w_m\}$ are not MTP2.

A rather general way to capture the stochastic dependence structure among random variables is the introduction of the concept of *multivariate copula* or, simply, *copula* (we refer to [10] for a detailed discussion). In particular, Sklar's Theorem [11] highlights how multivariate copulas model the dependence structure among random variables (see e.g. [10, Section 2.3]).

A popular family of copulas that found a number of applications is the Archimedean one, and [3] derive the conditions for an Archimedean copula to give rise to a MTP2 random vector. For the reader's convenience we recall here the definition of Archimedean copula:

Definition 2.3. *An Archimedean copula is a function* $C : [0,1]^m \to [0,1]$ *of the form*

$$C(\boldsymbol{u}) = \varphi^{-1}\left(\varphi(u_1) + \ldots + \varphi(u_m)\right) \quad \textit{for } u_i \in [0,1]$$

where the copula generator function $\varphi : [0,1] \to [0, +\infty]$ *is a strictly decreasing function with* $\lim_{t \to 0+} \varphi(t) = \infty$*,* $\varphi(1) = 0$*, and* φ^{-1} *is d-monotonic.*

An advantage of Archimedean copulas, that partly explain their theoretical and empirical success, is that they can represent a wide range of dependence properties, according to the specific generator function φ. An important feature of some classes of copulas, including the Archimedean one, is *exchangeability* (see e.g. [12]).

We recall here the definition of exchangeable copulas.

Definition 2.4. *The copula* $C : [0,1]^m \to [0,1]$ *is exchangeable if, for each* $\boldsymbol{u} \in [0,1]^m$ *and for each permutation* ϱ *of* $\{1,\ldots,m\}$*, one has:*

$$C(\boldsymbol{u}) = C(u_{\varrho(1)}, \ldots, u_{\varrho(m)}).$$

When Definition 2.4 is not satisfied, then copula C is said to be non-exchangeable.

Exchangeability implies symmetric dependence, which is typically modelled with Archimedean copulas by using just one or two parameters. This can be an undesirable property in practice. For this reason the development of new non-exchangeable copulas and the study of their properties are important fields of theoretical research.

3. Main result

The family of non-exchangeable copulas we deal with is generated by a one-dimensional function, and represents a generalization of the usual Archimedean copulas. We formalize it in the following:

Definition 3.1. *Fix* $J \in \mathbb{N}$ *and a set of* $m \times J$ *functions*

$$h_{jk} : [0,1] \to [0,1], \qquad j = 1,\ldots,J; \;\; k = 1,\ldots,m \tag{1}$$

such that:

(C3.1.i) h_{jk} *is differentiable in* $(0,1)$ *and strictly increasing in* $[0,1]$*, for all* j,k*;*

(C3.1.ii) $h_{jk}(0) = 0$ *and* $h_{jk}(1) = 1$*, for all* j,k*;*

(C3.1.iii) $\frac{1}{J}\sum_{j=1}^{J} h_{jk}(x) = x$, *for each* $k = 1, \dots, m$ *and* $x \in [0,1]$.

Moreover, define

$$\psi : [0,1] \to [0,1] \tag{2}$$

such that:

(C3.1.iv) ψ *is* $m+2$ *times differentiable in* $(0,1)$*;*

(C3.1.v) $\psi^{(i)} > 0$ *in* $(0,1)$*, for* $i = 1, \dots, m$*;*

(C3.1.vi) $\psi(0) = 0$ *and* $\psi(1) = 1$*.*

We define a non-exchangeable copula as $C_{NE}^{\psi} : [0,1]^m \to [0,1]$ *such that:*

$$C_{NE}^{\psi}(\boldsymbol{u}) = \psi^{-1}\left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right). \tag{3}$$

Remark 3.2. *We notice that copula* C_{NE}^{ψ} *is absolutely continuous (see* [8]*). The reference to function* ψ *will turn out to be useful for comparing the asymmetric copula in* (3) *with the usual Archimedean ones. However, the definition of the copula* C_{NE}^{ψ} *is achieved by employing several functions – the h's, specifically.*

Another important property of such a copula is that the case of an Archimedean copula is a sub-case of the setting proposed in (3)*. To show it, we first need to adopt the multiplicative representation of the Archimedean copulas (see* [8]*). Consider an Archimedean copula with generator* $\bar{\psi}$*. Then, we can write*

$$C^{\psi}(u_1, \dots, u_m) = \bar{\psi}^{-1}(\bar{\psi}(u_1) + \cdots + \bar{\psi}(u_m))$$

as

$$C^{\psi}(u_1, \dots, u_m) = \psi^{-1}(\psi(u_1) \times \cdots \times \psi(u_m)), \tag{4}$$

where $\psi(u) = \exp(-\bar{\psi}(u))$. *In so doing, the copula* C_{NE}^{ψ}, *defined in formula* (3), *becomes* C^{ψ} *in* (4) *if one takes* $J = 1$ *and* $h_{1k}(x) = x$, *for each* $k = 1, \ldots, m$ *and* $x \in [0, 1]$. *In this case* C_{NE}^{ψ} *is no longer non-exchangeable, and our theoretical framework becomes the symmetric case treated in* [3].

Copula (3) has been first introduced in [8, 9]. Nevertheless, the formulation presented here is different from that of the quoted papers. Specifically, as far as the copula's definition is concerned, conditions (C3.1.iv) and (C3.1.v) could be weakened. Indeed, [8, 9] propose only the m times differentiability of ψ and assume the less restrictive hypothesis that $\psi^{(i)} \geqslant 0$ in $(0, 1)$. However, our mildly stronger version is required to prove the main dependence result (see Theorem 3.3 below). Moreover, the proof of such a dependence result requires hypotheses involving jointly the behavior of the functions h_{jk} and ψ (see conditions (5) and (6) of Theorem 3.3). This outcome is due to the fact that ψ and h_{jk} are compounded in the definition of the copula C_{NE}^{ψ}, as equation (3) highlights.

Theorem 3.3. *Assume that the dependence among the components of the* m*-variate random vector* $\boldsymbol{W}$ *is described by copula* (3).

Furthermore, suppose that h_{jk} *is twice differentiable in* $(0, 1)$, *with*

$$\prod_{k=k_1,k_2} \Big[h''_{jk}(\psi(u_k)) \left(\psi'(u_k)\right)^2 + h'_{jk}(\psi(u_k))\psi''(u_k) \Big] \geqslant \Big[h''_{jk_1}(\psi(u_{k_1})) \times \\ \times \left(\psi'(u_{k_1})\right)^2 + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1}) \Big] \times h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2}) \qquad (5)$$

and

$$\prod_{k=k_1,k_2} h'_{jk}(\psi(u_k))\psi'(u_k) \geqslant \Big[h''_{jk_1}(\psi(u_{k_1})) \left(\psi'(u_{k_1})\right)^2 + \\ + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1}) \Big] \times h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2}) \,, \qquad (6)$$

holds for each $j = 1, \ldots, J,\ k_1, k_2 = 1, \ldots, m,\ k_1 \neq k_2$.

Suppose also that

$$\left(\psi^{-1}\right)^{(m+2)} \left(\psi^{-1}\right)^{(m)} - \left[\left(\psi^{-1}\right)^{(m+1)}\right]^2 \geqslant 0, \qquad \textit{in } (0,1). \tag{7}$$

Then $\boldsymbol{W}$ *is MTP2.*

Proof. By virtue of [3], it is sufficient to check that the density f of C_{NE}^{ψ} is log-supermodular, that is equivalent to saying that

$$\log(f(\boldsymbol{u})) := \log\left(\frac{\partial^m}{\partial u_1 \ldots \partial u_m} C_{NE}^{\psi}(\boldsymbol{u})\right) \tag{8}$$

is supermodular.

By (3) we have

$$\begin{aligned} f(\boldsymbol{u}) &= \frac{\partial^m}{\partial u_1 \ldots \partial u_m} C_{NE}^{\psi}(\boldsymbol{u}) \\ &= \left(\psi^{-1}\right)^{(m)} \left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right) \times \\ &\quad \times \frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\,. \end{aligned} \tag{9}$$

By (9) we can write

$$\begin{aligned} \log(f(\boldsymbol{u})) &= \log\left[\left(\psi^{-1}\right)^{(m)} \left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right)\right] + \\ &\quad + \log\left[\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\right] \\ &=: A(\boldsymbol{u}) + B(\boldsymbol{u})\,, \end{aligned} \tag{10}$$

where the terms $A(\cdot)$ and $B(\cdot)$ are an intuitive shorthand for the two $\log[\cdot]$ terms.

The supermodularity of $\log(f(\boldsymbol{u}))$ is equivalent to the following condition:

$$\frac{\partial^2}{\partial u_{k_1}\partial u_{k_2}}\left[A(\boldsymbol{u})+B(\boldsymbol{u})\right]\geqslant 0, \tag{11}$$

for each $k_1, k_2 \in \{1,\ldots,m\}$, and $(u_1,\ldots,u_m)\in[0,1]^m$. For an easier notation, we will pose hereafter

$$\xi := \frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k)). \tag{12}$$

We analyze the terms $A(\cdot)$ and $B(\cdot)$ separately.

First notice that

$$\frac{\partial A(\boldsymbol{u})}{\partial u_{k_1}} = \frac{\left(\psi^{-1}\right)^{(m+1)}(\xi)}{\left(\psi^{-1}\right)^{(m)}(\xi)}\times\frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1})\prod_{k\neq k_1}\left[h_{jk}(\psi(u_k))\right]$$

and

$$\begin{aligned}
\frac{\partial^2 A(\boldsymbol{u})}{\partial u_{k_1}\partial u_{k_2}} \quad = \quad & \frac{1}{\left\{\left(\psi^{-1}\right)^{(m)}(\xi)\right\}^2}\times\left\{\left(\frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1})\prod_{k\neq k_1}\left[h_{jk}(\psi(u_k))\right]\right)\times\right.\\
&\times\left(\frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_2}))\psi'(u_{k_2})\prod_{k\neq k_2}\left[h_{jk}(\psi(u_k))\right]\right)\times\\
&\times\left[\left(\psi^{-1}\right)^{(m+2)}(\xi)\times\left(\psi^{-1}\right)^{(m)}(\xi)-\left[\left(\psi^{-1}\right)^{(m+1)}(\xi)\right]^2\right]+\\
&+\left(\psi^{-1}\right)^{(m)}(\xi)\times\left(\psi^{-1}\right)^{(m+1)}(\xi)\times\\
&\times\frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1})h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2})\times\\
&\left.\times\prod_{k\neq k_1,k_2}\left[h_{jk}(\psi(u_k))\right]\right\}.
\end{aligned} \tag{13}$$

Hence, under Condition (C3.1.v) and hypothesis (7), we have

$$\frac{\partial^2 A(\boldsymbol{u})}{\partial u_{k_1}\partial u_{k_2}}\geqslant 0. \tag{14}$$

Let us now turn to $B(\cdot)$:

$$
\begin{aligned}
\frac{\partial B(\boldsymbol{u})}{\partial u_{k_1}} &= \frac{1}{\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)} \times \\
&\times\Big\{\sum_{j=1}^{J}\Big[h''_{jk_1}(\psi(u_{k_1}))\left(\psi'(u_{k_1})\right)^2 + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})\Big]\times \\
&\times \prod_{k\neq k_1} h'_{jk}(\psi(u_k))\psi'(u_k)\Big\}, \qquad (15)
\end{aligned}
$$

hence we have:

$$
\begin{aligned}
\frac{\partial^2 B(\boldsymbol{u})}{\partial u_{k_1}\partial u_{k_2}} &= \frac{1}{\Big[\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\Big]^2} \times \\
&\Bigg\{\Bigg[\sum_{j=1}^{J}\prod_{k\neq k_1,k_2} h'_{jk}(\psi(u_k))\psi'(u_k)\times \\
&\times \prod_{k=k_1,k_2}[h''_{jk}(\psi(u_k))(\psi'(u_k))^2 + h'_{jk}(\psi(u_k))\psi''(u_k)]\Bigg]\times \\
&\times \sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k) - \\
&- \Bigg[\sum_{j=1}^{J}\prod_{k\neq k_1} h'_{jk}(\psi(u_k))\psi'(u_k)\times \\
&[h''_{jk_1}(\psi(u_{k_1}))(\psi'(u_{k_1}))^2 + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})]\Bigg]\times \\
&\times \Bigg[\sum_{j=1}^{J}\prod_{k\neq k_2} h'_{jk}(\psi(u_k))\psi'(u_k)\times \\
&[h''_{jk_2}(\psi(u_{k_2}))(\psi'(u_{k_2}))^2 + h'_{jk_2}(\psi(u_{k_2}))\psi''(u_{k_2})]\Bigg]\Bigg\}. \qquad (16)
\end{aligned}
$$

By (16) we obtain that sufficient conditions for being $\partial^2 B_s(\boldsymbol{u})/(\partial u_{k_1}\partial u_{k_2}) \geqslant 0$ are given by relations in (5) and (6).

The result is proved, by the arbitrariness of k_1 and k_2. $\square$

It is possible to produce examples of generators of Archimedean copulas which, once substituted for ψ in the definition of copula C^{ψ}_{NE}, are such that $\boldsymbol{W}$ is not MTP2 by violating condition (C3.1.v) of Definition 3.1. Indeed, such a condition is crucial in the proof of Theorem 3.3 for the supermodularity of $\log(f(\boldsymbol{u}))$ in (8).

It is also important to point out that the conditions presented in Theorem 3.3 are the same of those presented in Theorem 2.11 of [3] when taking $J = 1$ and $h_{1k}(x) = x$. In fact, Theorem 2.11 of [3] states that the MTP2 property is equivalent to the log-convexity of $(-1)^m \left(\psi^{-1}\right)^{(m)}$, which is exactly formula (7). Furthermore, if $J = 1$ and $h_{1k}(x) = x$, then conditions (5) and (6) are trivially true.

Hence, by considering the multiplicative version of the Archimedean copulas as in Remark 3.2 and equation (4), the generator of the Clayton Archimedean copula is $\psi^C_\alpha(u) = \exp\left\{-\frac{1}{\alpha}(u^{-\alpha} - 1)\right\}$, that of the Frank copula is $\psi^F_\alpha(u) = \frac{e^{-\alpha u}-1}{e^{-\alpha}-1}$, while for the Gumbel case the generator is $\psi^G_\alpha(u) = \exp\left\{-(-\log(u))^\alpha\right\}$, where α is a parameter whose definition depends on the specific copula. In particular: $\alpha \in [-1, +\infty)\backslash\{0\}$ in the case of Clayton copula, $\alpha \in \mathbb{R}\backslash\{0\}$ in the Frank case and $\alpha \in [1, +\infty)$ in the Gumbel one.

In the Clayton case, we have

$$(\psi^C_\alpha(u))'' = \exp\left\{-\frac{1}{\alpha}(u^{-\alpha} - 1)\right\} \cdot u^{-2-\alpha}(u^{-\alpha} - \alpha - 1),$$

which is greater than zero not in the entire interval $(0, 1)$ but, rather, if and only if $u < (\alpha + 1)^{-1/\alpha}$.

For the Frank copula, we have

$$(\psi^F_\alpha(u))'' = \frac{\alpha^2 \exp\{-\alpha u\}}{\exp\{-\alpha\} - 1},$$

which is greater than zero if and only if $\alpha < 0$. So, also in this case there are some values of the parameter α leading to the violation of condition (C3.1.v).

The Gumbel copula case leads to

$$\begin{aligned}(\psi_\alpha^G(u))'' &= \exp\{-(-\log(u))^\alpha\} \cdot \alpha u^{-2}(-\log(u))^{\alpha-1} \times \\ &\quad \times \left[\alpha(-\log(u))^{\alpha-1} - (\alpha-1)(-\log(u))^{-1} - 1\right],\end{aligned}$$

which is greater than zero if and only if

$$\alpha(-\log(u))^{\alpha-1} - (\alpha-1)(-\log(u))^{-1} - 1 > 0.$$

Also the fulfillment of such an inequality is strongly related to the value of the parameter (for example, it is never satisfied when $\alpha = 1$).

We enter into some details for the Gumbel copula generator in the case of $\alpha = 1$. In this situation, we have $\psi_1^G(u) = \exp\{-(-\log(u))\} = u$. Copula in (3) becomes

$$C_{NE}^\psi(\boldsymbol{u}) = \frac{1}{J}\sum_{j=1}^J \prod_{k=1}^m h_{jk}(u_k). \tag{17}$$

For controlling the MTP2 property, it is sufficient to check that

$$\frac{\partial^2}{\partial u_{k_1}\partial u_{k_2}}\left\{\log\left(\frac{\partial^m}{\partial u_1 \dots \partial u_m}\left[\frac{1}{J}\sum_{j=1}^J \prod_{k=1}^m h_{jk}(u_k)\right]\right)\right\} \geqslant 0. \tag{18}$$

By (17), we can easily rewrite (18) as:

$$\frac{\sum_{j=1}^J \prod_{k\neq k_1,k_2} h'_{jk}(u_k) \times \prod_{k=k_1,k_2} h''_{jk}(u_k)}{\sum_{j=1}^J \prod_{k=1}^m h'_{jk}(u_k)} \geqslant 0, \ \ \forall\, k_1, k_2 = 1, \dots, m, \ \ k_1 \neq k_2 \tag{19}$$

which is not true in general and requires further conditions on the signs of the second derivatives of the functions h's. In fact, it is sufficient to consider two values $k_1 \neq k_2$ such that $h''_{jk_1}(u) < 0 < h''_{jk_2}(u)$, for each $u \in (0,1)$ and $j \in \{1, \dots, J\}$, for having that the left-hand side of (19) is less than zero.

The example of the Gumbel copula with $\alpha = 1$ is important also for deriving whether a complete monotone additive generator could lead to an asymmetric copula C^{ψ}_{NE} which satisfies the MTP2 property. In the present example, this is not the case. Indeed, according to Remark 3.2, we can write the additive generator of the Gumbel copula with $\alpha = 1$ by setting $\bar{\psi}(u) = -\log(u)$, for each $u \in [0,1]$. An easy computation gives that function $\bar{\psi}$ is completely monotone. However, further requirements on the h's must be satisfied to obtain the MTP2 property — see the comments on the validity of inequality (19).

To conclude, we can say that the non-exchangeable generalization of the Archimedean copulas with Clayton, Frank and Gumbel generator do not describe, in general, MTP2.

However, it is important to note that the set of copulas described by Definition 3.1 and Theorem 3.3 is not empty, and contains some cases of interest. We elaborate this point in the following.

Proposition 3.4. *Consider $J \times m$ positive real numbers $\alpha_{11}, \dots, \alpha_{1m}, \alpha_{21}, \dots,$ $\alpha_{2m}, \dots, \alpha_{Jm}$ such that*

$$\sum_{j=1}^{J} \alpha_{jk} = J.$$

Assume that the dependence among the components of the m-variate random vector $\boldsymbol{W}$ is described by copula

$$C^{\psi}_{NE}(\boldsymbol{u}) = \log\left[(e-1)\left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}\left(\frac{e^{u_k}-1}{e-1}\right)\right)+1\right], \qquad (20)$$

with

$$h_{jk}(x) = \begin{cases} \alpha_{jk}x, & \textit{for } x \in [0,1); \\ 1, & \textit{for } x = 1. \end{cases} \tag{21}$$

Then $\boldsymbol{W}$ *is MTP2.*

Proof. Copula in (20) is of the type described in (3), with

$$\psi(u_k) = \frac{e^{u_k} - 1}{e - 1}, \tag{22}$$

for each $k = 1, \ldots, m$.

Functions h's and ψ satisfy conditions (5), (6) and (7) of Definition 3.1 and (C3.1.i), (C3.1.iv), (C3.1.v) of Theorem 3.3. Moreover, by (21) and (22), it results

$$\frac{1}{J}\sum_{j=1}^{J} h_{jk}(x) = x, \qquad \forall\, k = 1, \ldots, m,\, x \in [0,1],$$

and $h_{jk}(0) = \psi(0) = 0$, $h_{jk}(1) = \psi(1) = 1$.

It is also easy to see that

$$\psi^{(i)}(x) = \frac{e^x}{e-1} > 0 \qquad \forall\, i = 1, \ldots, m,\, x \in (0,1),$$

and

$$\left(\psi^{-1}\right)^{(m+2)}(x)\left(\psi^{-1}\right)^{(m)}(x) - \left[\left(\psi^{-1}\right)^{(m+1)}(x)\right]^2 = 0, \qquad \forall\, x \in (0,1).$$

By definition of the h's, conditions (5) and (6) become

$$\psi'(x) = \psi''(x), \qquad \forall\, x \in (0,1),$$

which is trivially satisfied by function ψ in (22). Then, the h's and ψ satisfy the set of conditions listed in Definition 3.1 and the assumptions (5)–(7) of Theorem 3.3, and this gives the thesis. □

It is important to point out that copula in (20) describes a dependence of MTP2 type in a non-exchangeable setting. This turns out to be useful in several contexts. For example, it might find mathematical and statistical applications in reliability theory, when the components of the analyzed systems are modelled through heterogeneous random variables and show positive dependence (see e.g. [13] and the references therein): furthermore, it could also prove useful in multiple testing procedures in the presence of both positive and non-positive dependent test statistics (see e.g. [14]).

4. Concluding remarks

Multivariate total positivity of order 2 (MTP2, introduced in the literature by [1]) is an important multivariate dependence property with many applications in mathematics and statistics. In multivariate analysis, a relevant role is also played by the concept of copula (see e.g. [10]). In this paper we contribute to this field of research by studying the relationship between copulas and the MTP2 property in the case of non-exchangeability. In particular, we construct a rather wide family of non-exchangeable copulas that are associated to MTP2. The focus on non-exchangeability allows us to consider very flexible copulas, characterized by asymmetric dependence. However, we also show that our family of copulas, endowed with "classical" Archimedean copula generators (such as Gumbel, Clayton, and Frank), cannot generate MTP2.

Potential implications are far-reaching, involving different fields such as reliability theory and multiple testing.

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