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Sequential elimination in multi-stage all-pay auctions ☆

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ABSTRACT

I study a multi-stage all-pay auction in which the lowest bidder in each stage is eliminated. Elimination continues until only two bidders remain, one of whom wins the auction. I analyse optimal bidding behaviour and the seller's expected revenue when bidders have independent and private values. In contrast to typical bidding strategies, the optimal bid in each stage is strictly decreasing in the number of bidders. For a fixed number of bidders, however, bids increase as bidders progress through the stages of the auction. Despite independent values, this multi-stage auction yields less expected revenue to the seller than its single-stage counterpart when there are more than three bidders.

1. Introduction

All-pay auctions are a common approach to modelling competitive environments including lobbying (Baye et al., 1993), procurement (Fu et al., 2014) and crowdsourcing (Chawla et al., 2019). In many of these environments, the competition takes place over multiple stages and, in each stage, a number of bidders are eliminated. Moreover, it is natural to suppose that the value to a lobbyist from a bill being passed, or to a firm from procuring a contract, is private information. Much of the existing literature, however, has focused on settings in which these values are common knowledge.

Assuming that bidders' values are private and independent, I analyse a multi-stage all-pay auction in which the lowest bidder is eliminated in each stage. I first construct a model that is tractable, yet retains the ability to provide novel theoretical insights. I then characterise optimal bidding to understand how the number of bidders, and the stage in which each bidder places her bid, affects behaviour. Lastly, I calculate the seller's expected revenue to determine whether this multi-stage auction format outperforms its single-stage counterpart.

My results on optimal bidding show that, in contrast to several standard auction formats, the optimal bid in each stage decreases in the number of bidders. Intuitively, however, for a fixed number of bidders, each bidder's optimal bid increases as she progresses through the stages of the auction. Turning to the seller's expected revenue, there are three competing predictions one might have about this auction format. First, one might expect that, since there will be a larger number of bids in the multi-stage auction, expected revenue will be higher than in a single-stage all-pay auction.[1] Alternatively, one might think that, because bidders already shade their bids significantly in a single-stage all-pay auction, the addition of multiple elimination stages may increase the incentive to shade to an even greater degree. Therefore, expected revenue will be lower. Lastly, one might predict that, because the bidders' values are independent, the revenue equivalence theorem will hold. I show that, when there are more than three bidders, the second intuition is correct even when the bidders' values are independent. This result implies that the revenue equivalence theorem does not generally hold in this setting, and I discuss the mechanism that leads to this decrease in expected revenue.

A large literature analyses multi-stage all-pay auctions (e.g. Barut and Kovenock, 1998; Konrad and Kovenock, 2009; Sela, 2012; Hirata, 2014; Mendel et al., 2021). There are two main differences, however, between these papers and my model. First, they typically analyse the effects of multiple prizes. Second, and importantly, they assume complete information, which implies mixed-strategy equilibria. In contrast, I consider a setting in which only the winner in the final stage obtains the item and each bidder's value is private information. All-pay auctions with incomplete information have been investigated to a high degree of generality by, for example, Amann and Leininger (1996) and Krishna and Morgan (1997); however, these papers analyse single-stage auctions. An exception is Moldovanu and Sela (2006), who study a general two-stage all-pay auction under incomplete information. In

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[1] With four bidders, there will be nine total bids; with ten bidders, fifty-four bids.

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their model, however, the players' private information is related to ability, rather than their value for the prize.

Another strand of literature examines multi-stage Tullock contests with elimination (e.g. Clark and Riis, 1996; Yates and Heckelman, 2001; Arve and Chiappinelli, 2021). By employing the Tullock contest success function, these papers look at imperfectly discriminatory contests. While I study an all-pay auction, which is a perfectly discriminatory contest, I draw on this literature to support my assumptions of eliminating one participant per stage, awarding a prize only to the finalist, and concealing a bidder's relative ranking (Fu and Lu, 2012; Fu and Wu, 2022).

In Section 2, I lay out the details of the model. In Section 3, I analyse optimal bidding behaviour and the seller's expected revenue. I compare this expected revenue with the standard all-pay auction in Section 4. Proofs are in the Appendix.

2. Model

Consider a group $\mathcal{N} = \{1, 2, \dots, N\}$ of risk-neutral bidders who bid for an item in an all-pay auction with multiple elimination stages. Each bidder $i \in \mathcal{N}$ has an independent value $v_i \sim U[0,1]$ for the item, which is private information. There are $N-1$ stages and, in each stage, the bidder with the lowest bid is eliminated. Consequently, in stage 1, all N bidders take part in the auction whilst, in stage $N-1$, only two bidders remain. The bids from previous stages do not carry over, nor are they revealed. One can imagine bidders are simply informed they have made it to the next stage, without observing their opponents' bids or their own relative standing.

Each bidder solves for her optimal bidding strategy in each stage by backward induction. Therefore, in stage $N-1$, each of the two remaining bidders solve the standard all-pay auction maximisation problem: $\max_{b_i \in [0,1]} \Pr(b_i > b_j)v_i - b_i$. When each bidder uses the symmetric, increasing and continuous bidding strategy $\beta_{N-1} : [0,1] \to [0,1]$, this maximisation problem can be rewritten as $\max_{b_i \in [0,1]} \beta_{N-1}^{-1}(b_i)v_i - b_i$. The associated first-order condition is $(v_i/\beta'_{N-1}(\beta_{N-1}^{-1}(b_i))) = 1$. Using $\beta_{N-1}^{-1}(b_i) = v_i$ yields $\beta_{N-1}(v_i) = v_i^2/2$.

In stage $s = 1, \dots, N-2$, bidder i solves

$$\max_{b_i \in [0,1]} \beta_s^{-1}(b_i)\frac{v_i^{N-(1+s)}}{N+(1-s)} - b_i, \tag{1}$$

where $\beta_s : [0,1] \to [0,1]$ is symmetric, increasing and continuous in each stage s. The elements of (1) require some explanation. First, since bidder i only needs to avoid having the lowest bid in each stage to progress to the next, the term $\beta_s^{-1}(b_i)$ represents the probability of bidding higher than one other bidder in stage s. Second, given that the bidding strategy in stage $N-1$ is known to be $\beta_{N-1}(v_i)$, the probability of winning the auction in that stage depends only on the two remaining bidders' values. Therefore, the probability bidder i wins in stage $N-1$ is $\Pr(v_i > v_j | v_j > v_k \forall k \in \mathcal{N}/\{i,j\}) = 1/[N+(1-s)]$, which is increasing as bidder i progresses through the stages. Finally, using an order statistic argument, the probability of bidder i making it to stage $N-1$ is $v_i^{N-(2+s)}$. This term is the probability of bidder i having a higher value than at least one other bidder in all remaining stages, not including stage $N-1$. Multiplying this probability by v_i, the value obtained from winning the auction, yields the term $v_i^{N-(1+s)}$ in (1).

3. Optimal bidding and expected revenue

My first two results characterise the optimal bidding strategy in each stage.

Lemma 1. *The optimal bidding strategy in stage $s = 1, \dots, N-2$ is*

$$\beta_s(v) = \frac{v^{N-s}}{(N+1-s)(N-s)}.$$

Corollary 1. *$\beta_s(v)$ is decreasing in N and increasing in s for $v > 0$.*

Unlike the standard all-pay auction, each bidder's bid strictly decreases as the number of bidders increases. To see this, note that in the equivalent single-stage all-pay auction the optimal bid is $\beta(v,N) := [(N-1)/N]v^N$. For arbitrary $N = k$ and $N = k+1$, this bid is decreasing in the number of bidders if

$$\frac{\beta(v,k+1)}{\beta(v,k)} = \frac{\left(\frac{k}{k+1}\right)v^{k+1}}{\left(\frac{k-1}{k}\right)v^k} = \left[\frac{k^2}{(k+1)(k-1)}\right]v < 1.$$

This expression simplifies to $v < \bar{v} := 1 - (1/k^2)$, which, for fixed $k \geq 2$, does not hold for all v. Therefore, bids in the single-stage all-pay auction are increasing in the number of bidders if $v \in (\bar{v}, 1]$. This difference in bidding behaviour, as a function of the number of bidders, is illustrated in Fig. 1.[2]

More intuitively however, for a fixed number of bidders, each bidder's bid increases as she progresses through the stages of the auction. In early stages, the probability of both making it to the final stage and winning the auction are low. Therefore, bidders initially shade their bids relatively severely. As they move through the stages, though, the probability of winning increases, so the degree of shading decreases.

Using $\beta(v,N)$, the optimal bid in the single-stage all pay auction, the seller's expected revenue can be written as $\mathbb{E}_v[\Pi_A(N)] := N\mathbb{E}_v[\beta(v,N)] = N\mathbb{E}_v\left[(N-1)v^N/N\right] =$

$$(N-1)\mathbb{E}_v\left[v^N\right] = (N-1)\int_0^1 v^N dv = (N-1)/(N+1).$$

The expected revenue in the all-pay auction with elimination stages is

$$\begin{aligned}\mathbb{E}_v[\Pi_E(N)] &:= N\mathbb{E}_v[\beta_1(v)] + (N-1)\mathbb{E}_v[\beta_2(v)] + \cdots + 3\mathbb{E}_v[\beta_{N-2}(v)] \\ &\quad + 2\mathbb{E}_v[\beta_{N-1}(v)],\end{aligned}$$

which, using Lemma 1, can be written as

$$\begin{aligned}\mathbb{E}_v[\Pi_E(N)] &= \frac{1}{N-1}\mathbb{E}_v\left[v^{N-1}\right] + \frac{1}{N-2}\mathbb{E}_v\left[v^{N-2}\right] + \cdots \\ &\quad + \frac{1}{N-(N-2)}\mathbb{E}_v\left[v^{N-(N-2)}\right] + \mathbb{E}_v\left[v^2\right] \\ &= \sum_{n=N-1}^{N-(N-2)} \frac{1}{n}\mathbb{E}_v\left[v^n\right] + \mathbb{E}_v\left[v^2\right] = \sum_{n=2}^{N-1}\frac{1}{n}\mathbb{E}_v\left[v^n\right] + \mathbb{E}_v\left[v^2\right] \\ &= \sum_{n=2}^{N-1}\frac{1}{n}\int_0^1 v^n dv + \int_0^1 v^2 dv \\ &= \sum_{n=2}^{N-1}\frac{1}{n(n+1)} + \frac{1}{3}.\end{aligned}$$

The sum term in the expected revenue can be simplified as

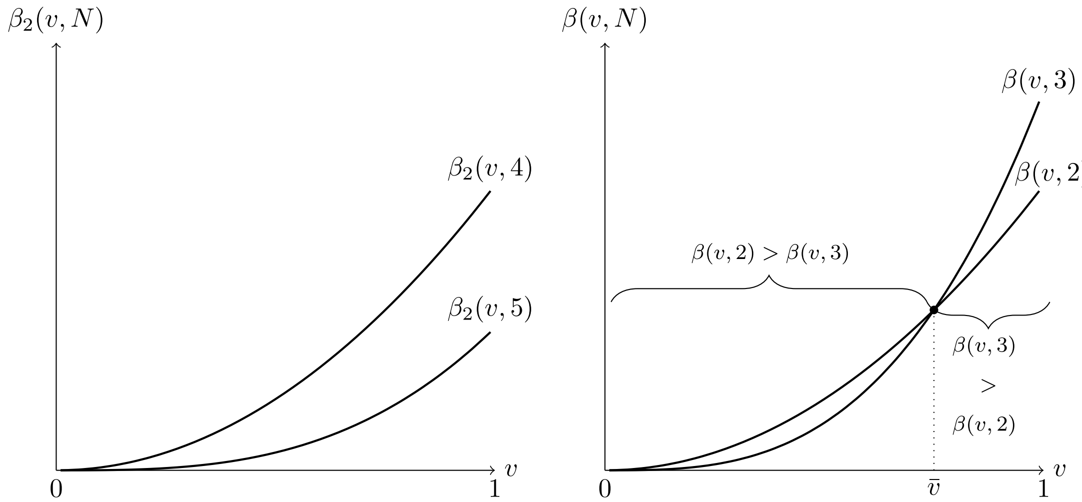
$$\sum_{n=2}^{N-1}\frac{1}{n(n+1)} = \sum_{n=2}^{N-1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2} - \frac{1}{N}. \tag{2}$$

The first step comes from partial fraction decomposition and the second comes from noting that (2) is a telescoping sum. Despite each bidder's bid decreasing in the number of bidders, my next result shows that the seller's expected revenue is still increasing as N becomes larger.

Lemma 2. *$\mathbb{E}_v[\Pi_E(N)]$ is increasing in N.*

Lemma 2 implies that the aggregate effect of a larger number of bidders is relatively greater than the individual reduction in the bids each bidder optimally submits.

[2] Bidding behaviour in the multi-stage all-pay auction is also unlike the first-price auction, where the optimal bid is $v^{1-N}\beta(v,N) = [(N-1)/N]v$. This bid is increasing in N as $v^{1-N}\beta(v,k+1) = [(k+1-1)/(k+1)]v > [(k-1)/k]v = v^{1-N}\beta(v,k)$ is equivalent to $k^2 > (k+1)(k-1) = k^2 - 1$.



(a) Multi-stage auction bids in $s = 2$. (b) Single-stage auction bids.

Fig. 1. Optimal bids as a function of the number of bidders.

4. Expected revenue comparison

My main result demonstrates the effect that elimination stages have on the seller's expected revenue in comparison to a single-stage all-pay auction.

Proposition 1. $\mathbb{E}_v[\Pi_A(N)] > \mathbb{E}_v[\Pi_E(N)]$ *for* $N > 3$.

Perhaps surprisingly, the all-pay auction with elimination stages can yield less expected revenue than the standard all-pay auction. Therefore, revenue equivalence does not hold when there are more than three bidders, which implies more than two stages in the auction. Despite the bidders' values being independently distributed, remaining in multiple stages of the auction gives implicit information regarding the other bidders' values for the item. When a bidder makes it to stage $N-1$, for example, they know that their opponent has the highest value of the other $N-2$ bidders, which is evidence that their opponent's value is large. Then, knowing that their opponent's value is likely to be large, the bidder's conditional belief about their winning probability changes. These effects cause bidders to shade to such a degree that the overall expected revenue is less than in a standard all-pay auction with the same number of bidders.

When there are exactly three bidders, however, the all-pay auction with elimination stages yields the same expected revenue as the standard all-pay auction. Therefore, it also yields the same expected revenue as the first-price, and second-price, auction with three bidders.

Corollary 2. $\mathbb{E}_v[\Pi_A(3)] = \mathbb{E}_v[\Pi_E(3)]$.

When there are three bidders, the effects of the conditional probability of winning in the final stage are small enough to not affect bidders' behaviour. The result also arises because, with three bidders, there are no intermediate stages. In particular, the seller's expected revenue with three bidders can be written as

$$\mathbb{E}_v[\Pi_A(3)] = 2\int_0^1 v^3 dv = 3\int_0^1 \frac{v^2}{6} dv + 2\int_0^1 \frac{v^2}{2} dv = \mathbb{E}_v[\Pi_E(3)].$$

When the number of bidders increases to four, for example, the expected revenue in the single-stage and multi-stage auction is $\mathbb{E}_v[\Pi_A(4)] = 3\int_0^1 v^4 dv$ and

$$\begin{aligned}\mathbb{E}_v[\Pi_E(4)] &= 4\int_0^1 \frac{v^3}{12} dv + 3\int_0^1 \frac{v^2}{6} dv + 2\int_0^1 \frac{v^2}{2} dv \\ &= \frac{1}{3}\int_0^1 v^3 dv + \mathbb{E}_v[\Pi_E(3)],\end{aligned}$$

respectively. The expected revenue in the single-stage all-pay auction is greater as

$$\begin{aligned}\mathbb{E}_v[\Pi_A(4)] - \mathbb{E}_v[\Pi_A(3)] = 3\int_0^1 v^4 dv - 2\int_0^1 v^3 dv > \\ \frac{1}{3}\int_0^1 v^3 dv = \mathbb{E}_v[\Pi_E(4)] - \mathbb{E}_v[\Pi_E(3)].\end{aligned}$$

When the number of bidders increases to four in the multi-stage auction, there are two effects. First, the total number of bids increases from five to nine. Whilst this increase in total bids increases expected revenue, there is an underlying trade-off: more bidders implies more elimination stages. As the number of stages increase, each bidder's degree of shading in the first stage also increases. The second effect is that the bids placed in the second stage decrease. These relatively lower bids arise from the decrease in the probability of both making it to the final stage and winning the auction. The degree of shading in the first stage, and the adverse effect on bids placed in the second stage, is sufficient for the sole extra bidder in the single-stage auction to yield greater expected revenue. Moreover, both effects are amplified as more bidders are added. With five bidders, bids placed in stages two and three decrease. The combination of highly shaded bids in the first stage, and the decrease in bids placed in intermediate stages when the number of bidders increases, is what causes revenue equivalence to break down when there are more than three bidders.

To generalise the above argument, define $\Delta\mathbb{E}_v[\Pi_{A-E}(N)] := \mathbb{E}_v[\Pi_A(N)] - \mathbb{E}_v[\Pi_E(N)]$ as the difference in expected revenue between the two auction formats.

Corollary 3. $\Delta\mathbb{E}_v[\Pi_{A-E}(N+1)] > \Delta\mathbb{E}_v[\Pi_{A-E}(N)]$ *for* $N \geq 3$.

As the number of bidders increases, the difference in expected revenue between the single-stage and multi-stage all-pay auction also increases. For example, with four bidders, the gain in terms of expected revenue from the standard all-pay auction is $1/60$, whilst for five bidders it is $1/30$. However, this pattern of doubling does not hold generally. The difference in expected revenue between the two auction formats comprises a Cauchy sequence with a limit of $1/6$.

Declaration of competing interest

The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Proofs

Proof of Lemma 1. The first-order condition associated with (1) is

$$\frac{1}{\beta_s'(\beta_s^{-1}(b_i))}\frac{v_i^{N-(1+s)}}{N+(1-s)} = 1.$$

Using $\beta_s^{-1}(b_i) = v_i$ and integrating yields $\beta_s(v_i) = [1/(N+1-s)]\int v_i^{N-(1+s)}dv_i = [1/((N+1-s)(N-s))]v_i^{N-s}$. □

Proof of Corollary 1. Fix $v > 0$ and let

$$\beta(N,s) := \frac{v^{N-s}}{(N+1-s)(N-s)}.$$

Since $N \in \mathbb{N}/\{1,2\}$ and $s = 1,\dots,N-2$, varying N whilst keeping s fixed, and vice versa, yields the sequences $\{\beta(3,s),\beta(4,s),\dots\}$ and $\{\beta(N,1),\beta(N,2),\dots,\beta(N,N-2)\}$, respectively. To show the former is decreasing in N, take arbitrary $N = k$ and $N = k+1$ and note that

$$\begin{aligned}\frac{\beta(k+1,s)}{\beta(k,s)} &= \frac{\frac{v^{k+1-s}}{(k+2-s)(k+1-s)}}{\frac{v^{k-s}}{(k+1-s)(k-s)}} = \frac{v^{k+1-s}}{v^{k-s}}\frac{(k+1-s)(k-s)}{(k+2-s)(k+1-s)}\\ &= \frac{v(k-s)}{k+2-s} < 1.\end{aligned}$$

To show the latter is increasing in s, take arbitrary $s = k$ and $s = k+1$ and note that

$$\begin{aligned}\frac{\beta(N,k+1)}{\beta(N,k)} &= \frac{\frac{v^{N-k-1}}{(N-k)(N-k-1)}}{\frac{v^{N-k}}{(N+1-k)(N-k)}} = \frac{v^{N-k-1}}{v^{N-k}}\frac{(N+1-k)(N-k)}{(N-k)(N-k-1)}\\ &= \frac{N+1-k}{v(N-k-1)} > 1.\end{aligned}$$ □

Proof of Lemma 2. Using (2), the expected revenue in the all-pay elimination auction is

$$\mathbb{E}_v[\Pi_E(N)] = 5/6 - 1/N,$$

which is increasing in N. □

Proof of Proposition 1. Define

$$\begin{aligned}\Delta\mathbb{E}_v[\Pi_{A-E}(N)] :=& \mathbb{E}_v[\Pi_A(N)] - \mathbb{E}_v[\Pi_E(N)] = \frac{N-1}{N+1} - \left(\frac{5}{6} - \frac{1}{N}\right)\\ =& \frac{(N-2)(N-3)}{6N(1+N)},\end{aligned}$$

and note that $\Delta\mathbb{E}_v[\Pi_{A-E}(N)] > 0$ for all $N > 3$. □

Proof of Corollary 2. When $N = 3$, $\mathbb{E}_v[\Pi_A(3)] = 1/2 = \mathbb{E}_v[\Pi_E(3)]$. □

Proof of Corollary 3. To show that $\Delta\mathbb{E}_v[\Pi_{A-E}(N)]$ is increasing in N for $N \geq 3$, take arbitrary $N = k$ and $N = k+1$ and note that

$$\begin{aligned}\Delta\mathbb{E}_v[\Pi_{A-E}(k+1)] - \Delta\mathbb{E}_v[\Pi_{A-E}(k)] &= \frac{(k-1)(k-2)}{6(k+1)(k+2)} - \frac{(k-2)(k-3)}{6k(k+1)}\\ &= \frac{(k-2)[k(k-1)-(k-3)(k+2)]}{6k(k+1)(k+2)}\\ &= \frac{k-2}{k(k+1)(k+2)} > 0.\end{aligned}$$ □

References

Amann, E., Leininger, W., 1996. Asymmetric all-pay auctions with incomplete information: The two-player case. Games Econom. Behav. 14 (1), 1–18.
Arve, M., Chiappinelli, O., 2021. The role of budget constraints in sequential elimination tournaments. Scand. J. Econ. 123 (4), 1059–1087.
Barut, Y., Kovenock, D., 1998. The symmetric multiple prize all-pay auction with complete information. Eur. J. Political Econ. 14 (4), 627–644.
Baye, M.R., Kovenock, D., de Vries, C.G., 1993. Rigging the lobbying process: An application of the all-pay auction. Am. Econ. Rev. 83 (1), 289–294.
Chawla, S., Hartline, J.D., Sivan, B., 2019. Optimal crowdsourcing contests. Games Econom. Behav. 113, 80–96.
Clark, D.J., Riis, C., 1996. A multi-winner nested rent-seeking contest. Public Choice 87 (1/2), 177–184.
Fu, Q., Jiao, Q., Lu, J., 2014. Disclosure policy in a multi-prize all-pay auction with stochastic abilities. Econom. Lett. 125 (3), 376–380.
Fu, Q., Lu, J., 2012. The optimal multi-stage contest. Econom. Theory 51, 351–382.
Fu, Q., Wu, Z., 2022. Disclosure and favoritism in sequential elimination contests. Am. Econ. J. Microecon. 14 (4), 78–121.
Hirata, D., 2014. A model of a two-stage all-pay auction. Math. Social Sci. 68, 5–13.
Konrad, K.A., Kovenock, D., 2009. Multi-battle contests. Games Econom. Behav. 66 (1), 256–274.
Krishna, V., Morgan, J., 1997. An analysis of the war of attrition and the all-pay auction. J. Econom. Theory 72 (2), 343–362.
Mendel, M., Pieroth, F., Seel, C., 2021. Your failure is my opportunity—Effects of elimination in contests. J. Math. Econom. 95, 102495.
Moldovanu, B., Sela, A., 2006. Contest architecture. J. Econom. Theory 126 (1), 70–96.
Sela, A., 2012. Sequential two-prize contests. Econom. Theory 51, 383–395.
Yates, A.J., Heckelman, J.C., 2001. Rent-setting in multiple winner rent-seeking contests. Eur. J. Political Econ. 17 (4), 835–852.