FOURIER-SERIES-BASED VIRTUAL FIELDS METHOD FOR THE IDENTIFICATION OF 2-D STIFFNESS DISTRIBUTIONS

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ABSTRACT: The Virtual Fields Method (VFM) is a powerful technique for the calculation of spatial distributions of material properties from experimentally-determined displacement fields. A Fourier-series-based extension to the VFM (the F-VFM) is presented here, in which the unknown stiffness distribution is parameterised in the spatial frequency domain rather than in the spatial domain as used in the classical VFM. We summarise here the theory of the F-VFM for the case of elastic isotropic thin structures with known boundary conditions. An efficient numerical algorithm based on the 2-D Fast Fourier Transform reduces the computation time by 3-4 orders of magnitude compared to a direct implementation of the F-VFM for typical experimental dataset sizes. Reconstruction of stiffness distributions with the F-VFM has been validated on several stiffness distribution scenarios, one of which is presented here, in which a difference of about 0.5% was achieved between the reference and recovered stiffness distributions.

1. INTRODUCTION

Inverse problems may arise in solid mechanics when there is significant unknown variation in the spatial distribution of the material properties, i.e., of the constitutive equations, or the parameters in those equations. Finite element model updating (FEMU) is one method to solve problems of this type, by adjusting an approximate finite element model until the responses it produces are as close to those acquired from experiments as possible.

An alternative approach is the virtual fields method (VFM). The development of the VFM was inspired by a relevant interpretation of the equation of the virtual work principle [1]. The advantage of the VFM compared to other methods is its ability to solve inverse problems of this type without any iteration. Numerous applications of the VFM to date can be found in [1]. The key feature in any application of the VFM is the selection of the virtual fields. Several techniques based on different choices of virtual fields have been presented [2-5]. The common point is the selection of the virtual fields as polynomials of spatial variables (either on the whole domain or in a piecewise form), and the material properties are considered as having single values (homogeneous) within the domain. The first attempt to parameterise the material properties as a function of spatial variables was proposed in [6] where the authors tried to reconstruct the spatial-dependent stiffness map of a plate with impact damage.

In this paper, we retain the basic concepts underlying the VFM but approach the parameterisation of the material properties in the spatial frequency, rather than spatial, domain by performing a 2-D Fourier series expansion of the stiffness distribution over the region of interest. Furthermore, the virtual fields are not selected as polynomials of spatial variables as in the previous VFM literature, but from a set of simple cosine or sine functions of different spatial frequencies. The VFM with a Fourier series for the material property parameterisation and cosine/sine functions for the virtual fields will be denoted the F-VFM. An example of the successful application of the F-VFM to the identification of an unknown stiffness distribution under known boundary conditions is summarised here; further details are given in [7].

2. THEORETICAL

2.1 Virtual Fields Method formulation

For a thin 2-D sample made of an isotropic material, subject to known traction distributions around its boundary and negligible volume forces, the fundamental equation underlying the VFM may be written as follows [1,7]:

$$\int_S \left(\left(\varepsilon_{xx}^* \varepsilon_{xx} + \varepsilon_{yy}^* \varepsilon_{yy} + \frac{1}{2} \varepsilon_{ss}^* \varepsilon_{ss} \right) + \nu \left(\varepsilon_{xx}^* \varepsilon_{xx} + \varepsilon_{yy}^* \varepsilon_{yy} - \frac{1}{2} \varepsilon_{ss}^* \varepsilon_{ss} \right) \right) Q_{xx} dS = \int_\ell \left(T_x u_x^* + T_y u_y^* \right) d\ell \tag{1}$$

where S is the area of interest within the structure over which the experimental data are available; ℓ is the part of the boundary of S on which tractions exist; t is the thickness of the sample; ε_{xx} and ε_{yy} are the normal strains as measured along the x- and y-axes of a Cartesian coordinate system, and ε_{ss} is the engineering shear strain; (T_x, T_y) are the components of the traction vector defined over ℓ; and u_x^* and u_y^* are the components of the virtual displacement field with ε^*_{xx}, ε^*_{yy} and ε^*_{ss} the corresponding virtual strain field components. The unknown stiffness distribution $Q_{xx}(x,y)$ is related to Young's modulus and Poisson's ratio ν (assumed here to be a constant) by $Q_{xx} = E/(1-\nu^2)$.

2.2 Fourier series expansion of stiffness distribution

The stiffness distribution $Q_{xx}(x,y)$ in Eqn. (1) may be expanded as a 2-D Fourier series and written in matrix form as follows:

$$Q_{xx}(x,y) = (1 \;\ldots c_{m,n}(x,y) \ldots \ldots s_{m,n}(x,y) \ldots) \begin{pmatrix} a_{0,0} \\ . \\ a_{m,n} \\ . \\ b_{m,n} \\ . \end{pmatrix} \tag{2}$$

where the shorthand notation $c_{m,n}(x,y) = \cos 2\pi\left(\frac{mx}{L_x} + \frac{ny}{L_y}\right)$ and $s_{m,n}(x,y) = \sin 2\pi\left(\frac{mx}{L_x} + \frac{ny}{L_y}\right)$ is used for cosine and sine functions with non-dimensional spatial frequency components (m, n), where m, n = 0,1,2,…,N. The column vector on the right hand side of Eqn. (2) is the sought solution vector, consisting of $(N+1)^2$ $a_{m,n}$ coefficients associated with the cosine functions and $(N+1)^2 - 1$ $b_{m,n}$ coefficients associated with the sine functions. The total number of unknown Fourier coefficients in Eqn. (2), i.e. the number of degrees of freedom in the identification problem, is therefore

$$N_F = 2(N+1)^2 - 1 \;. \tag{3}$$

2.3 Selection of virtual fields in the F-VFM

The natural choice for the virtual fields in the F-VFM is an arrangement of simple cosine and sine functions. Eqn. (1) involves area integrals of terms of the form $\varepsilon^*_{\alpha\alpha}\varepsilon_{\beta\beta}Q_{xx}$ (α, β = x, y, s); the use of different spatial frequencies in the virtual fields therefore allows a given spatial frequency in the measured strain field $\varepsilon_{\beta\beta}$ to be linked in turn with different coefficients in the Fourier expansion of Q_{xx}. No special optimised fields have been developed yet for the F-VFM, but a few simple rules have been used to select the virtual fields as follows:

1. The set of virtual field spatial frequencies should be the same as that for the modulus parameterization so that a given spatial frequency in the measured strain field $\varepsilon_{\beta\beta}$ can be linked in turn with all the coefficients in the Fourier expansion of Q_{xx};
2. Each spatial frequency for a given virtual strain field component should have both a sine and cosine wave of unit amplitude to ensure that the signal in $\varepsilon_{\beta\beta}$ at that spatial frequency is detected regardless of its phase;
3. The total number of virtual fields should be equal to N_F in order to determine uniquely the unknown Fourier series coefficients of Eqn. (2).

The approach taken here was therefore to define a set of fields with both u^*_x and u^*_y chosen to produce the unit waves in ε^*_{xx} and ε^*_{yy} simultaneously. Thus both the ε^*_{xx} and ε^*_{yy} fields consist of a set of cosine waves (with spatial frequency components (p, q) where p, q = 0,1,…,N, giving $(N+1)^2$ independent virtual fields), and a set of corresponding sine waves (in which the trivial case $p = q = 0$ is excluded, giving an additional $(N+1)^2 - 1$ fields). The total number of chosen cosine and sine virtual fields will therefore be equal to N_F, which is the required number to determine uniquely the unknown Fourier series coefficients of Eqn. (2). The required virtual shear strain field can be computed from the u^*_x and u^*_y fields defined in the above way.

Substituting these virtual fields into Eqn. (1) results in the matrix equation

$$\mathbf{MX} = \mathbf{Y} \tag{4}$$

where **M** is an $N_F \times N_F$ matrix whose elements are calculated from the experimental strain fields, **X** is the desired $N_F \times 1$ solution vector of Fourier coefficients, and **Y** is an $N_F \times 1$ column vector calculated from the tractions. Eqn. (4) can be inverted by the MATLAB `pinv` function.

2.4 Fast Fourier VFM implementation

When applying Eqn. (1) to experimental data, the measured strain fields ε_{xx}, ε_{yy} and ε_{ss} are normally sampled on a regular grid and the integrals are replaced by summations. If the experimental strain fields have $N_x \times N_y$ pixels, then a single contributory term to one of the elements of **M** requires a minimum of $N_x N_y$ addition plus multiplication operations. The computational effort to calculate **M** therefore scales as $N_F^2 N_x N_y$. For example, the application given in the next section, with $N_x = N_y = 1000$ and $N_F = 881$ ($N = 20$), took approximately 5.5×10^3 s to set up the **M** matrix on an Intel® Core™ i7 CPU 2.79 GHz machine with 8GB of memory.

A much more efficient algorithm can be implemented, however, using 2-D fast Fourier transforms. The fact that both the expansion of Q_{xx} and the virtual fields are represented as sine and cosine functions means that the integrals can be expressed as 2-D Fourier coefficients of a linear combination of the experimental strain fields. It can be shown [7] that a total of only four 2-D Fast Fourier Transforms (FFTs) are required to assemble all the terms in **M**. The computational effort for each 2-D FFT is of order $N_x N_y(\log_2(N_x) + \log_2(N_y))$ operations, whereas that for assembling the elements of **M** from the resulting coefficients is of order $N_F^2 \sim 2N^4$ operations. The latter dominates over the former for problems involving relatively large numbers of Fourier coefficients in the reconstruction. In such cases, the computational effort becomes essentially independent of the resolution of the experimental strain fields, with a theoretical reduction in computational effort by a factor of $N_x N_y$ by using the fast algorithm over the direct (i.e., element by element) method of assembling the matrix **M**.

The computation time for the other steps in the algorithm, i.e. evaluation of the terms in the vector **Y**; the inversion of Eqn. (4); and reconstruction of the elastic stiffness distribution from the solution vector, is normally short compared to that for calculation of **M**. The reconstruction can be handled very efficiently by performing a single 2-D inverse Fourier transform on a 2-D array of complex numbers that is obtained directly from **X**.

For the problem considered in the next section, the total calculation time for the stiffness identification using the fast algorithm when implemented as a MATLAB script, on an Intel® Core™ i7 CPU 2.79 GHz machine with 8GB of memory, was ~2.5 s and 250 s for problem sizes $N = 20$ and $N = 80$, respectively. This may be compared with values of 6.1×10^3 s and 3.7×10^6 s, respectively, for the direct method. A time saving of 3-4 orders of magnitude is therefore clearly achievable in practice.

3. EXAMPLE APPLICATION

In this section we give proof-of-principle results from the F-VFM method presented above with a complex stiffness distribution under uniform loading conditions. The input data to the F-VFM method were provided by a forward calculation from known stiffness distributions by the finite element method, thus providing a benchmark to compare the reconstructed stiffness maps against. The FE model used to generate the input strain fields consisted of a thin square plate of size $L_x \times L_y = 10 \times 10$ mm^2 and of thickness $t = 1$ mm. The geometry was meshed in Mentat2010 using 1000×1000 linear quadrilateral elements with full integration. Two vertical edges of the plate perpendicular to the *x*-axis were loaded with a uniformly distributed stress $\sigma_{xx} = 1$MPa pointing outwards. The origin of the coordinate system is at the centre of the plate (see Fig. 1).

The material was chosen to be linear elastic isotropic with the reference elastic modulus distribution given by an ‘eggbox’ pattern of spatially varying stiffness distribution

$$E = 20 + \cos 2\pi\left(\frac{2x}{L_x} + \frac{y}{L_y}\right) + \sin 2\pi\left(\frac{x}{L_x} + \frac{2y}{L_y}\right), \tag{5}$$

and a constant Poisson’s ratio $\nu = 0.3$. Plane stress conditions are applicable in this case since the thickness of the plate is relatively small compared to the other dimensions.

Some of the main results are shown in Fig. 2. Ripples in the recovered stiffness map are observed but can be largely removed by smoothing with a uniform square kernel of size equal to the pitch p (= 50 pixels) of the highest frequency fringes. The edge effect region of 25 pixels wide (half of the kernel window size) resulting from the convolution is masked out from the reconstructed stiffness as in Fig. 2(c). The residual in the error map (Fig. 2(d)) represents a difference of about 0.5% between the reference and recovered stiffness distributions.

4. CONCLUSIONS

The paper presents a development of the virtual fields method by implementing a novel parameterisation of the stiffness distribution with a full 2-D Fourier series expansion. An example stiffness distribution has been reconstructed after a single computation step without any iteration. A highly efficient numerical algorithm based on the fast Fourier Transform allows an identification problem with ~10^3 degrees of freedom to be solved in just a few seconds. The spatial resolution of the recovered stiffness by F-VFM is directly controllable through the choice of maximum spatial frequency. In this study the reconstructed modulus fields were obtained under the assumption that the traction distributions are known over the boundaries. In the future the F-VFM will be extended to cope with the cases where boundary conditions are unspecified over at least a part of the boundary of the domain of interest. Other important remaining issues are application of the method to experimental strain fields (as opposed to strain fields calculated by a forward finite element analysis), and extension to the case of anisotropic materials such as carbon-fibre reinforced composites.

5. REFERENCES

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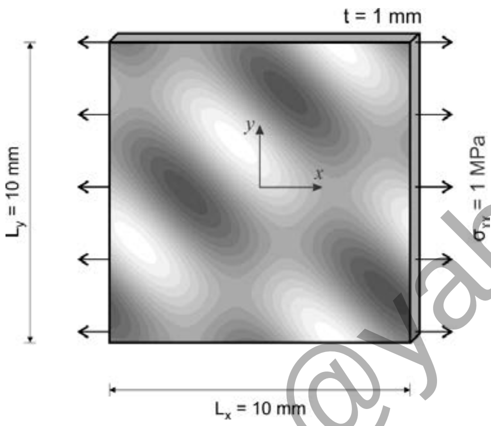

Figure 1 – A square plate of 'eggbox' stiffness distribution subject to horizontal uniform stress σ_{xx}.

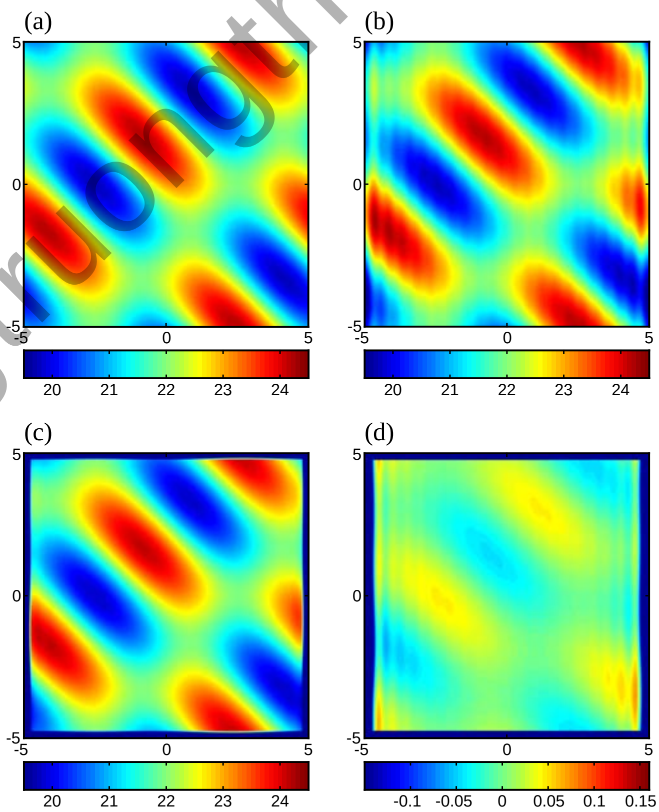

Figure 2 – Recovery of the 'eggbox' stiffness distribution from Figure 1 by F-VFM with *N* = 20 (881 degrees of freedom, units: MPa). (a) Reference stiffness map (1000×1000 pixels); (b) Recovered stiffness map by F-VFM; (c) as (b) after smoothing by a 50×50 pixel kernel; (d) Error map.