Microeconomic modeling of financial time series with long term memory

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Abstract: *In this paper we fix a microeconomic model of exchange rates and we give the explicit relation between model's parameters and its long memory properties. This avoids long numerical calibration procedures and allows to build the model with the parameters suitable for the required long memory degree.*

Keywords: *Microeconomic, model, long memory property*

1 INTRODUCTION

The presence of long memory property of financial data has been evidenced through several papers. These analyses concern time series of shares' prices, price increments, returns, and several functions of returns (absolute returns, squared returns, powered returns). However, microeconomic explanation of these data often is not obvious.

In this paper we provide a microeconomic model for long memory time series. The main property of this model is the functional relation between its parameters and the long memory parameters of the time series under examination. This allows an immediate calibration of the model avoiding time-expensive numerical calibration procedures.

We start from the statements of the models introduced by Kirman [22], [23], and by Kirman and Teyssiere [24] . In their paper [24] the authors assume that the market exchange rates are determined by the interaction of several agents that act on the market driven by two different opinions: the fundamentalists and the chartists. The forecasts made from these two groups are due to different analyses of the market data. We modify the evolution of the agents' opinions given in [24] transferring the distinction between fundamentalists and chartists into the decision procedure of each agent. Each of them is not purely fundamentalist or purely chartist any more, but follows a mixed strategy, depending on the influence of different sources of information. Moreover, we introduce a new term that allows each agent to perform a self-correction on his own forecasts.

The paper is organized as follows: for convenience of self references of the paper the next paragraph resumes the main definitions and properties that will be used in the following paragraphs. Section 3 contains the description of our model and Section 4 provides the proof of the long memory properties. Section 5 contains a short description of the model of Kirman and Teyssiere and Section 6 explores the differences between the two models.

2 DEFINITIONS AND PROPERTIES

2.1 Long term memory

A stationary process $\{X_t\}$ is called stationary process with long memory if its autocorrelation function $\rho(k)$ has asymptotically the following hyperbolic rate of decay:

$$\rho(k) \sim L(k)k^{2d-1} \ \ ak \to \infty$$

where $L(k)$ is a slowly varying function, i. e. $L(\lambda k)/L(k) \to 1$ as $k \to \infty$, $\forall \lambda > 0$.

The parameter d summarizes the degree of long range dependence of the series. If $-0.5 < d < 0$ the series is *mean reverting*; if $d = 0$ there is no correlation between the data and, if $0 < d < .5$ the correlation function decays slowly with the lag k and the time series has a *long range* or long memory property.

Instead, the autocorrelation function of short memory processes decay to zero at an exponential rate [19] [24].

2.2 Hurst's H exponent

Given a time series $\{X_t\}$ Hurst's exponent H describes the degree of dependence among the increments of the analyzed process. It can be defined as follows:

$$E(X_{t+\tau} - X_t)^2 \sim c\tau^{2H}$$

Several methods are available for its estimate [25] [24] and $H = d + \frac{1}{2}$.

2.3 Long term memory and spectral analysis estimate

Spectral analysis can provide an estimate for H. The spectral density of a covariance stationary time series

$\{X_t\}$ is given by

$$f(\lambda) = \gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h)cos(\lambda h)$$

where $\gamma(h) = Cov(X_t, X_{t-h})$ is the autocovariance function.

The spectrum of stationary processes with long range memory can be approximated in the neighborhood of the zero frequency as

$$S(f) \propto f^{-\alpha},\ 1 < \alpha < 3, f \to 0^+$$

The following relation holds: $H = \frac{\alpha - 1}{2}$. ([28], [26])

Any covariance stationary time series with hyperbolically decreasing autocovariance function of the form $\rho(h) \sim h^{2d-1}$ with $0 < d < .5$ is a long memory process, i.e. the decay of the autocorrelation function uniquely determines the size of the process' long memory. If $d = 0$ then $\{X_t\}$ is a short memory process.

2.4 Long term memory and I(d) processes

A time series $\{X_t\}$ is called fractionally integrated with differencing parameter d ($X_t \sim I(d)$) , if

$$X_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}\ with\ c_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}$$

$$and \epsilon_t \sim i.i.d.(0, \sigma^2)$$

d is called the fractional degree of integration of the process and $H = d + \frac{1}{2}$ [15],[34]

2.5 Beta distribution and its properties

Definition 2.5.1 The random variable z is an ordinary beta-distributed if its probability density function is defined as

$$p(z) = \frac{1}{B(a,b)} z^{a-1}(1-z)^{b-1},\ 0 < z < 1, \qquad (1)$$

where a e b are positive parameters and $B(a, b)$ is the beta function defined by

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du$$

This distribution is referred to as $b(a, b)$.

Proposition 2.5.1
If $X \sim b(a, b)$, then the random variable $Y = 1 - X$ is a beta generalized random variable with law $b(b, a)$.

Let us now consider a new random variable X which is related to Z through the power transformation

$$Z = (\frac{X}{C})^h \text{ or } X = CZ^{\frac{1}{h}} \qquad (2)$$

The parameter h may be either positive or negative. By this transformation we can define a generalization of the beta distribution.

By $b(p, q, h, C)$ we denote the four real parameters distribution generalized beta, defined by the density function f given by

$$f(x) = \frac{|h|}{CB(p,q)}(\frac{x}{h})^{ah-1}[1 - (\frac{x}{C})^h]^{b-1},\ 0 \le x \le C.$$

Definition 2.5.2 The random variable x defined by (2) has a beta generalized distribution if its probability density function is defined by

$$b(a,b,C,h;x) = \frac{|h|}{B(a,b)C}(\frac{x}{C})^{ah-1}[1 - (\frac{x}{C})^h]^{b-1} \quad (3)$$

where $0 \le x \le C$.

The moment M_n of order n for X is given by

$$M_n = C^n \frac{B(a+\frac{n}{h}, b)}{B(a,b)} = C^n \frac{\Gamma(a+b)\Gamma(a+\frac{n}{h})}{\Gamma(a+b+\frac{n}{h})\Gamma(a)}. \quad (4)$$

Remark 2.5.1 A standard beta random variable is also a generalized beta random variable with parameters $h = C = 1$. Thus the properties of the beta standard random variable can be extended to the beta generalized random variable.

The beta generalized distribution is close with respect to the class of power transformations.

Proposition 2.5.2 Let $X \sim b(a, b, C, h)$. The random variable

$$Y = rX^s, \qquad (5)$$

where $r, s \in$. Then $Y \sim b(a, b, rC^s, \frac{h}{s})$.

Remark 2.5.2 Given $X \sim b(a, b, C, h)$. From the last proposition follows that,

$$\lambda X \sim b(a, b, \lambda C, h)$$

and

$$X^\eta \sim b(a, b, C^\eta, \frac{h}{\eta})$$

3 MICROECONOMIC MODEL

In this paper we propose a microeconomic description for financial time series with long term property. In particular, we consider the application of the model for the description of exchange rates.

Exchange rates are determined by the interaction of several small investors. There are no market makers. Let us consider only one foreign currency. The microeconomic explanation of the long memory property enters the decision procedure of each agent, whose final action is driven by several causes: thus we don't provide a neat distinction between the pure fundamentalists and the pure chartists. We describe the situation in which each agent tries to get information from different sources: technical analysis and observation of the fundamentals. Other information sources like observation of the behavior of the other agents, social interaction, hierarchical interaction, external shocks here are not considered. Thus, in order to make a forecast $\Delta P_{i,t+1}|I_{i,t}$ of the exchange rate increment $\Delta P_{i,t+1}$ conditioned to the information available at time t, $I_{i,t}$, each agent i relies on a technical analysis forecast $\Delta P^c_{i,t+1}|I_{i,t}$ conditioned to his information at time t, and on fundamentalist forecast $\Delta P^f_{i,t+1}|I_{i,t}$, conditioned to his information at time t, too. Let us indicate the individual proportion between the two points of view. Thus

$$(\Delta P_{i,t+1}|I_{i,t}) = k_i(\Delta P^f_{i,t+1}|I_{i,t}) + (1-k_i)(\Delta P^c_{i,t+1}|I_{i,t}),$$

The exchange rate of the market is given by the average of the exchange rates associated to the agents, i.e.

$$P_t = \sum_{i=1}^{N} \frac{1}{N} P_{i,t}. \qquad (6)$$

The chartist approach assumes that the investor can get information by observing the time series of the market data. In this model we consider chartist forecast composed by two terms: for the sake of simplicity, a forecast due to the increment of market exchange rates made by using the simplest linear model

$$h_{t-1}(P_t - P_{t-1})$$

where h is constant, plus an additive term,

$$\bar{\alpha}_i(P_{t-1} - P_{i,t-1})$$

where $\bar{\alpha}_i \in D[0,1]$, $\forall\, i$, that takes into account a self correction of the agent obtained by the observation of the difference between the previous market price and the previous agent forecast. Thus we have that the chartist forecast is given by

$$\Delta P^c_{i,t+1}|I_{i,t} = h_{t-1}(P_t - P_{t-1}) + \bar{\alpha}_i(P_{t-1} - P_{i,t-1}) \quad (7)$$

and

$$\begin{aligned}(1-k_i)\Delta P^c_{i,t+1}|I_{i,t} &= (1-k_i)h_{t-1}(P_t - P_{t-1}) + \quad (8)\\ +(1-k_i)\bar{\alpha}_i(P_{t-1} - P_{i,t-1}) &= (1-k_i)h_{t-1}P_t +\\ +(1-k_i)(\bar{\alpha}_i - h_{t-1})P_{t-1} &- (1-k_i)\bar{\alpha}_i P_{i,t-1}\end{aligned}$$

So we have a linear relation between the exchange rate predicted at time $t+1$ and the variation of P_t, independent from the agent, and we have an additional stochastic shock associated to the comparison between the market situation at time $t-1$ and the forecast made by the agent at the same date.

In the fundamentalist analysis the value of the market fundamentals is known, and so the investor has a complete information on the estimate of the exchange rate (he understands if the exchange rate is over or under estimated). We thus have the following relation:

$$\Delta P^f_{i,t+1}|I_{i,t} = \nu(\tilde{P}_{i,t} - P_t), \qquad (9)$$

where $\tilde{P}_{i,t}$ is a series of fundamentals observed with a stochastic error from the agent i at time t, i.e.

$$\tilde{P}_{i,t} = \bar{P}_{i,t} + \alpha_{i,t}$$

with $\alpha_{i,t} = \beta_{i,t}P_t$ and $\beta_{i,t} \in D[0,1]$.
The fundamental variables $\bar{P}_{i,t}$ can be described by the following random walk:

$$\bar{P}_{i,t} = \bar{P}_{i,t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2_\epsilon)$$

Thus

$$k_i \Delta P^f_{i,t+1}|I_{i,t} = k_i \nu \bar{P}_{i,t} + k_i \nu(\beta_{i,t} - 1)P_t, \qquad (10)$$

We suppose furthermore that each agent may invest in foreign riskly value with stochastic interest rate $\rho_t \sim N(\rho, \sigma^2_t)$ and in riskless bonds with a constant interest rate r. We have to suppose that $\rho > r$ (otherwise we are in meaningless hypothesis).
Let us define with $d_{i,t}$ the demand of the foreign value associated to agent i at the date t. Thus the wealth invested in foreign riskly value is given by $P_{t+1}d_{i,t}$ and, taking into account the stochastic interest rate ρ_{t+1}, we have that the wealth grows as $(1+\rho_{t+1})P_{t+1}d_{i,t}$. The remaining part of the wealth, $(W_{i,t} - P_t d_{i,t})$ is invested in riskless bonds and thus gives $(W_{i,t} - P_t d_{i,t})(1+r)$. The wealth of the agent i at time $t+1$ is given by $W_{i,t+1}$, and it can be written by

$$W_{i,t+1} = (1+\rho_{t+1})P_{i,t+1}d_{i,t} + (W_{i,t} - P_{i,t}d_{i,t})(1+r).$$

The expression of $W_{i,t+1}$ can be rewritten as

$$\begin{aligned}W_{i,t+1} = (1+\rho_{t+1})\Delta P_{i,t+1}d_{i,t} + W_{i,t}(1+r) - \quad (11)\\ -(r - \rho_{t+1})P_{i,t}d_{i,t}.\end{aligned}$$

The utility function associated to the agent i, and conditioned to his information at time t, is defined by:

$$U(W_{i,t+1}|I_{i,t}) = E(W_{i,t+1}|I_{i,t}) - \mu V(W_{i,t+1}|I_{i,t}),$$

where E and V are the usual mean and variance operators and are given by:

$$E(W_{i,t+1}|I_{i,t}) = (1+\rho)\Delta P_{i,t+1}d_{i,t} + W_{i,t}(1+r) - (r-\rho)P_{i,t}d_{i,t}$$

and

$$V(W_{i,t+1}|I_{i,t}) = V[(1+\rho_{t+1})P_{i,t+1}](d_{i,t})^2.$$

Each agent i can change his demand $d_{i,t}$ in order to maximize the expected utility, conditioned to his information at the date t.

For each agent i the first order condition is

$$(1+\rho)\Delta P_{i,t+1} - (r-\rho)P_{i,t} - 2\mu V[(1+\rho_{t+1})P_{i,t+1}]d_{i,t} = 0,$$

and thus we obtain

$$d_{i,t} = b_{i,t}P_{i,t} + g_{i,t}\Delta P_{i,t+1},$$

with

$$b_{i,t} = \frac{\rho - r}{2\mu V(P_{i,t+1}(1+\rho_{t+1}))},$$

$$g_{i,t} = \frac{\rho + 1}{2\mu V(P_{i,t+1}(1+\rho_{t+1}))}.$$

Let $X_{i,t}$ be the supply of foreign value for the agent i. When the market is in equilibrium, the interest rate, that is used by the investor for the transactions, is such that

$$X_{i,t} = b_{i,t}P_{i,t} + g_{i,t}\Delta P_{i,t+1}.$$

thus $-X_{i,t}/b_{i,t} = -P_{i,t} - (g_{i,t}/b_{i,t})\Delta P_{i,t+1}$. Continuing to follow the Kirman and Teyssiere approach we assume that

$$\bar{P}_{i,t} = \frac{X_{i,t}}{b_{i,t}}.$$

Setting $c = -(b_{i,t}/g_{i,t}) = \frac{1+\rho}{r-\rho}$ we have that

$$\Delta P_{i,t+1} = -c\bar{P}_{i,t} + cP_{i,t}$$

By these relations we obtain:

$$P_{i,t} = (\nu k_i/c - 1)\bar{P}_{i,t} + \frac{1}{c}[(\beta_{i,t}-1)\nu k_i + (1-k_i)h_{t-1}]P_t + \tag{12}$$

$$+\frac{1}{c}[(1-k_i)(\bar{\alpha}_i - h_{t-1})]P_{t-1} - \frac{1}{c}(1-k_i)\bar{\alpha}_i P_{i,t-1}$$

4 LONG MEMORY PROPERTY OF THE MICROECONOMIC MODEL

By the definition of P_t given by (6), we have the following result:

Proposition Suppose that the following conditions hold:

1. $\beta_{i,t} = -\frac{h_{t-1}}{\nu k_i} + \frac{h_{t-1}}{\nu} + 1$;
2. $\bar{\alpha}_i = (1-k_i)^{\delta}$
3. $k_i \sim b(p, p, 1, 1)$,
4. $p \in (-1, 1)$.

Then, for $N \to +\infty$, we have that P_t has long memory with Hurst exponent given by $H = \frac{p+1}{2}$.

Proof Let L be the difference operator such that $LP_{i,t} = P_{i,t-1}$.
Define $\hat{\beta}_i = -\frac{1}{c}(1-k_i)\bar{\alpha}_i$, $\hat{\alpha}_i\hat{\epsilon}_{t-1} = \frac{1}{c}(1-k_i)(\bar{\alpha}_i - h_{t-1})$.

From the hypothesis $k_i \sim b(p,p,1,1)$ and from proposition 2.5.1 follows that $(1-k_i) \sim b(p,p,1,1)$. From this result and by applying proposition 2.5.2 follows that $(1-k_i)(-\frac{1}{c}) \sim b(p,p,-\frac{1}{c},1)$. For particular $\bar{\alpha}_i$ we have that $\hat{\beta}_i$ still has a beta distribution. As example this happens if $\bar{\alpha}_i = (1-k_i)^{\delta}$. In this case $\hat{\beta} \sim b(p,p,-\frac{1}{c},\delta)$.

Then (12) becomes (by using also the first hypothesis):

$$P_{i,t} = \frac{(1/c)\nu k_i + 1}{1-\hat{\beta}_i L}\bar{P}_{i,t} + \frac{\hat{\alpha}_i\hat{\epsilon}_{t-1}}{1-\hat{\beta}_i L}P_{t-1}. \tag{13}$$

For the definition of P_t and $\bar{P}_{i,t}$, we can write

$$P_t = \sum_{i=1}^{N}\frac{1}{N}[\frac{(1/c)\nu k_i + 1}{1-\hat{\beta}_i L}\bar{P}_{i,t} + \frac{\hat{\alpha}_i\hat{\epsilon}_{t-1}}{1-\hat{\beta}_i L}P_{t-1}]. \tag{14}$$

In the limit for $N \to \infty$ and by the definition of $\bar{P}$ we have

$$P_t = E[\frac{(1/c)\nu k_i + 1}{1-\hat{\beta}_i L}\bar{P}_{i,t}] + E[\frac{\hat{\alpha}_i\hat{\epsilon}_{t-1}}{1-\hat{\beta}_i L}P_{t-1}] =$$

$$P_t = \sum_{k=1}^{\infty}P_{t-k}\hat{\epsilon}_{t-k}\int_0^1 \frac{\hat{\alpha}}{(1-\hat{\beta}L)}dF(\hat{\alpha},\hat{\beta})$$

Suppose, as a further hypothesis, that there exist a random variable $\alpha^* \sim D(0,1)$ with mean μ such that $\hat{\alpha} = (1-\hat{\beta})\alpha^*$, and α^* is independent from $\hat{\beta}$. Thus

$$P_t = \sum_{k=1}^{\infty}P_{t-k}\hat{\epsilon}_{t-k}\int_0^1 \alpha^*(1-\hat{\beta})\hat{\beta}^{k-1}dF(\alpha^*,\hat{\beta}) =$$

$$= \sum_{k=1}^{\infty}P_{t-k}\hat{\epsilon}_{t-k}\int_0^1 \alpha^* dF(\alpha^*)\int_0^1 (1-\hat{\beta})\hat{\beta}^{k-1}dF(\hat{\beta}) =$$

$$=: \sum_{k=1}^{\infty}a_k P_{t-k}\hat{\epsilon}_{t-k}. \tag{15}$$

Thus

$$a_k = c_1\frac{B(p+k-1,p-1)}{B(p,p)} \sim c_2 k^{-1-p} \tag{16}$$

This is a characteristic of a long memory process [16]. Thus we have a long memory model I(d) with $d = p$ and thus Hurst exponent $H = p + \frac{1}{2}$ ([8], [9], [16], [17], [19], [20], [25]).

5 THE KIRMAN AND TEYSSIERE MODEL

The microeconomic model introduced by Kirman and Teyssiere in [24] analyzes the exchange rates and it uses an epidemiological model developed by Kirman ([22], [23]). Their basic idea is the existence of two groups of agents, called chartists and fundamentalists, who differ by their price forecast. The important feature of these models is that individuals change from being fundamentalists and become chartists and vice-versa, but at time t each agent behaves either purely as a chartist or purely as a fundamentalist. Thus, the groups are not fixed in size and this has consequences for market behavior. The model is an equilibrium one, i.e. the level of the exchange rate at each time point is such that the demand and supply are equal. By assuming very short time intervals, this can be a realistic hypothesis. In the model, it has been assumed that the market is not efficient, and the agents can predict the price at time $t+1$, given the information at time t, by the following relation:

$$E(P_{t+1}|I_t) = \Delta P_{t+1}|I_t + P_t, \qquad (17)$$

where $\Delta P_{t+1}|I_t$ is the predicted price change at time $t+1$, given the information set I_t. Let P_t be the exchange rate at time t. Chartists make the assumption, that the next change in exchange rate is a linear function of the previous price change, i.e.

$$\Delta P^c_{t+1}|I_t = h(P_t - P_{t-1}) = h\Delta P_t, \qquad (18)$$

So we have a linear relation between the exchange rate predicted at time $t+1$ and the variation of P_t, for each agent that takes position in the market. On the other hand, the fundamentalist analysis is based on the known value of the market fundamentals, and so the investors understand if the exchange rate is over or under estimated. We have the following relation:

$$\Delta P^f_{t+1}|I_t = \nu(\tilde{P}_{i,t} - P_t), \qquad (19)$$

where $\tilde{P}_{i,t}$ is a series of fundamentals observed with a stochastic error, i.e. $\tilde{P}_t = \bar{P}_t + \alpha_t$, with $\alpha_t \sim N(0, \sigma^2_\alpha)$. In the model it has been assumed that the fundamentals $\bar{P}_t$ follow a random walk given by:

$$\bar{P}_t = \bar{P}_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2_\epsilon)$$

Denote by ρ_t the foreign interest rate, $d_{i,t}$ the demand by the i^{th} individual for foreign currency and r the domestic interest rate. The exchange rate P_t and the foreign interest rate ρ_t are considered by the agents as independent random variables, with $\rho_t \sim N(\rho, \sigma^2_t)$ and $\rho > r$.
A part of the wealth $W_{i,t}$ of the agent i at time t is invested in foreign currency, and the remaining part in the domestic value. At time $t+1$, by the effect of the interest rates, the cumulated wealth of the agent i, $W_{i,t+1}$, is given by

$$W_{i,t+1} = (1+\rho_{t+1})P_{t+1}d_{i,t} + (W_{i,t} - P_t d_{i,t})(1+r).$$

The expression of $W_{i,t+1}$ can be rewritten as

$$W_{i,t+1} = (1+\rho_{t+1})\Delta P_{t+1}d_{i,t} + W_{i,t}(1+r) - (r-\rho_{t+1})P_t d_{i,t}. \qquad (20)$$

The utility function associated to the agent i, and conditioned to his information at time t, is defined by:

$$U(W_{i,t+1}|I_{i,t}) = E(W_{i,t+1}|I_{i,t}) - \mu V(W_{i,t+1}|I_{i,t}),$$

where E and V are the usual mean and variance operators and are given by:

$$E(W_{i,t+1}|I_{i,t}) = (1+\rho)\Delta P_{t+1}d_{i,t} + W_{i,t}(1+r) - (r-\rho)P_t d_{i,t}$$

and

$$V(W_{i,t+1}|I_{i,t}) = V[(1+\rho_{t+1})P_{t+1}](d_{i,t})^2,$$

and μ is a constant, that denotes the risk adversion coefficient.
In order to maximize the expected utility, each agent i changes his demand $d_{i,t}$, conditioned to his information at the date t.
The first order condition is

$$(1+\rho)\Delta P_{t+1} - (r-\rho)P_t - 2\mu V[(1+\rho_{t+1})P_{t+1}]d_{i,t} = 0,$$

and thus we obtain

$$d_{i,t} = b_{i,t}P_t + g_{i,t}\Delta P_{t+1},$$

with

$$b_{i,t} = \frac{\rho - r}{2\mu V(P_{t+1}(1+\rho_{t+1}))},$$
$$g_{i,t} = \frac{\rho + 1}{2\mu V(P_{t+1}(1+\rho_{t+1}))}.$$

Let X_t be the supply of foreign value. The condition such that the market is in equilibrium is that the supply for foreign exchange X_t is equal to the demand, i.e.

$$X_t = b_t P_t + g_t \Delta P_{t+1},$$

where we split the market forecast ΔP_{t+1} into the part forecasted by the fundamentalists and the part forecasted by the chartists, i.e.

$$\Delta P_{t+1} = k_t \Delta P^f_{t+1} + (1-k_t)\Delta P^c_{t+1}, \qquad (21)$$

where k_t is the proportion of agents making a forecast based on a fundamentalist approach.
Let us denote

$$\bar{P}_t = \frac{X_t}{b_t}.$$

By these relations we obtain:

$$P_t = c[\nu k_t(\tilde{P}_t - P_t) + (1-k_t)h(P_t - P_{t-1})] + \bar{P}_t, \qquad (22)$$

by assuming that $\bar{P}_t = X_t/b_t$, where the constant c is given by

$$c = \frac{1+\rho}{r-\rho}.$$

Keeping into account the definition of P_t, the (22) can be written as

$$P_t = \frac{1}{1-\gamma_t}P_{t-1} - \frac{\gamma_t}{1-\gamma_t}\bar{P}_t + \frac{ck_t\nu}{1+c(k_t\nu - (1-k_t)h)}\alpha_t \qquad (23)$$

where

$$\gamma_t = \frac{1+ck_t\nu}{c(1-k_t)\nu}$$

(22) and (23) are valid under the condition that the denominator is different from zero, i.e.

$$1 + c(k_t\nu - (1-k_t)h) \neq 0 \qquad (24)$$

or, equivalently, if γ_t is different from one. Since k_t is the only unknown parameter in equation (24), given that $c(\nu + h) \neq 0$, then $1 + ck_t\nu - c(1 - k_t)h = 0$ if and only if

$$k_t = \frac{ch - 1}{c(\nu + h)}.$$

Given that $k_t \in [0, 1]$, then $1 + ck_t\nu - c(1 - k_t)h \neq 0$ if and only if

$$\frac{ch - 1}{c(\nu + h)} > 1 \text{ or } \frac{ch - 1}{c(\nu + h)} < 0. \qquad (25)$$

Given that c is negative, the conditions in (25) are equivalent to

$$\frac{ch - 1}{c(\nu + h)} > 1 \Leftrightarrow \nu < \frac{\rho - r}{1 + \rho}. \qquad (26)$$

If $\nu = 0$, then the fundamentalists believe that markets are efficient, and the condition (26) is satisfied.

Now we want to analyse the process governing the evolution of k_t. The assumptions of the autors are that agents interact and agents comunicate their beliefs on the next period forecast through a particular epidemiologic process introduced by Foellmer in [13]. Since the parameters of the epidemiologic model are independent of the previous parameters of the model, the proportion of fundamentalists and the forecasts of the agent are independent of the economic variables. Let θ_t be the number of agents with a fundamentalist forecast at time t. An assumption of the model is that pairs of agents meet themselves at random and the probability that the first agent is converted to the opinion of the second one is equal to $(1 - \delta)$. Moreover, each agent change by himself, independently, his opinion with probability ζ. The dynamic evolution of θ_t is given by a Markov chain. We can write that θ becomes [23]

$$P(\theta, \theta + 1) = (1 - \frac{\theta}{N})[\epsilon + (1 - \delta)\frac{\theta}{N - 1}]$$

$$P(\theta, \theta - 1) = \frac{\theta}{N}[\epsilon + (1 - \delta)\frac{N - \theta}{N - 1}]$$

$$P(\theta, \theta) = 1 - P(\theta, \theta + 1) - P(\theta, \theta - 1)$$

After the meeting, the proportion of fundamentalists is equal to θ_t/N. In the Kirman and Teyssiere model, proposed in [24], the conditional probabilities are also affected by the following rule:

$$k_t = \frac{\sum_{i=1}^{N} \mathbf{1}_{(k_{i,t} > 0.5)}}{N}, \qquad (27)$$

where $\mathbf{1}$ is the indicator function and $k_{i,t}$ is the observation at time t by the agent i of this proportion, and it is affected by a stochastic error, i.e.

$$k_{i,t} = \frac{\theta_t}{N} + \epsilon_{i,t}, \text{ with } \epsilon_{i,t} \sim N(0, \sigma_\theta^2).$$

If agent i observe that $k_{i,t} > 0.5$, then he will make a fundamentalist forecast, otherwise he will make a chartist forecast. Thus the (27).

6 COMPARISON BETWEEN THE MODELS

The model that we have introduced can be reduced to the Kirman and Teyssere [24] model for particular values of the parameters. This comparison let understand the role of the parameters of the models for the description of long memory time series.

In the limit $N \to \infty$ and if $\bar{\alpha}_{i,t} \equiv 0, \forall i, t$ the model introduced in paragraph 3 produces the same dynamics for P_t of the Kirman and Teyssiere [24] model with $\sigma_\theta = 0$.

Let us compare the evolution equations of P_t for the two models. The initial description of the agents' behavior is different in the two models, but for some values of the parameters it is possible to obtain the same evolution equation.

Keeping into account the results reported in [24] we have that the introduction of $\bar{\alpha}_{i,t}$ terms is strictly necessary for the discussion of the long term memory properties of the model.

The hypothesis that k_i are beta variable is not strictly necessary, but it is useful in order to get a comparison of the structure that produces the k_i. In the limit $N \to \infty$ instead of the markovian evolution of k_i we have that the random variable k obeys a $b(\alpha, \alpha, 1, 1)$, distribution [22],[23]. Thus the hypothesis in our model of k_i beta is useful in order to keep the comparison between the two models.

7 CONCLUSIONS

In the model proposed in [24], the long term memory property has been analyzed by various model independent tests, by a numerical implementation point of view. The main results are around the returns (that are uncorrelated) and absolute returns and squared returns (that display long memory). The microeconomic model presented in this work is more general, and the Kirman and Teyssiere model can be seen as a particular case of the one presented in this paper. Anyway, our aim is different. We prove that the exchange rates, under certain hypothesis, can be represented as stochastic processes with long term memory property.

The introduced model can be extended in several ways. As an example $\bar{\alpha}_i$ can have different distributions, as long as the long memory properties continue to hold. It could be interesting to fix the largest class that provides the same results. The distribution of k_i has been assumed to be a Beta distribution in order to get comparable the evolution equations of P_t. We didn't introduce any further explanation for this distribution as due to the behavior of the agents, like the epidemiological model. Thus it could be interesting to explore the possibility to introduce a more detailed interaction dynamics for k_i that allows to get the same long memory properties of the model.

Another generalization concerns the distribution of

k_i. If we are not interested any more to keep on the comparison with the model of Kirman and Teyssiere [24],[22],[23] then we can change distribution in such a way that the proof of long memory properties continues to hold.

The model are both equilibrium models. Another interesting generalization could concern the application of the equilibrium model for time series different from the exchange rates ones, as example dealing with riskly and not riskly assets with stochastic interest rates.

Thus there are several interesting possible developments, last but not least a deeper analysis of the used hypotheses of the distribution of variables in order to fix the minimum amount of hypotheses that it is necessary to use.

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