A Spatial Mixed Poisson Framework for Combination of Excess of Loss and Proportional Reinsurance Contracts

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Abstract

In this paper a purely theoretical reinsurance model is presented, where the reinsurance contract is assumed to be simultaneously of an excess of loss and of a proportional type. The stochastic structure of the set of pairs (claim's arrival time, claim's size) is described by a Spatial Mixed Poisson Process. By using an invariance property of the Spatial Mixed Poisson Processes, we estimate the amount that the ceding company obtains in a fixed time interval in force of the reinsurance contract.

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JEL Classification: C65, G22

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1 Introduction

Insurance and reinsurance models have been the focus of a good part of actuarial research, and the interest on this topic is still growing.

Several approaches to describe reinsurance and to solve related optimization problems have been attempted in the actuarial literature, based on risk theory, economic game theory and stochastic dynamic control. The literature on these subjects is almost endless. Examples of research in each of these directions are the papers by Dickson and Waters (1996, 1997), Centeno (1991, 1997), Krvavych (2001) for risk theory; by Aase (2002), Suijs et al. (1998) for economic game theory; by Schmidli (2001, 2002), Hipp and Vogt (2001) and Taksar and Markussen (2003) for stochastic dynamic control.

In the field of insurance and reinsurance, Mixed Poisson processes are widespread both in the literature and in the applications. Mixed Poisson processes on the line and infinite-server queue models are very well known and used (see e.g. Grandell (1997)); but also a spatial setting can be useful in modelling several practical situations (e.g. spatial queues, see e.g. Cinlar (1995)).

We present here a case where a spatial setting turns out to be useful. Actually, we consider a reinsurance model based on Spatial Mixed Poisson Processes (SMPP, hereafter). Our perspective is purely theoretical, and we develop our model by using stochastic techniques grounded on invariance of the family of such processes under infinite-server queue-type transformations.

Reinsurance-type contracts are basically of two different kinds: *proportional reinsurance* and *excess of loss reinsurance*. In the proportional, or "pro rata" reinsurance, the reinsurer indemnifies the ceding company for a predetermined portion of the losses. In the case of excess of loss, or "non-proportional" reinsurance, on the contrary, the reinsurer indemnifies the ceding company for all losses or for a specified portion of them, but only if the claims' sizes fall within a prespecified band.

We are interested in the case when the reinsurance contract is simultaneously of a proportional and of a excess-of-loss type.

The combination of excess of loss and proportional reinsurance has been in fact widely used to construct reinsurance models. Centeno (1985) proposed a statistical combination, searching for the optimal one, by using three moment functions of the insurer's retained risk. In Schmitter (1987) the optimal linear combination between the two types of reinsurance has been determined, as a constrained optimization problem. Hurlimann (1994a-c) focuses on the hedge properties of a mixed proportional-excess of loss reinsurance contract. Other papers in this field are Centeno (1986, 2002), Kaluszka (2001), Schmitter (2001) and Verlaak and Beirlant (2003), who also consider optimal solutions for a quota share-excess of loss combination.

A common feature of the most part of the quoted papers is that optimality coincides with the minimization of the ruin probability of the insurance company. The claims' sizes are assumed to be independent and identically distributed, and the arrival times to follow a Poisson process.

The ceding company is hereafter denoted by $\mathcal{C}_{\mathcal{A}}$ and the reinsurer company by $\mathcal{C}_{\mathcal{B}}$.
In the present paper, we will deal with the estimate of the amount that $\mathcal{C}_{\mathcal{A}}$ obtains from $\mathcal{C}_{\mathcal{B}}$ in a fixed time interval, based on the information collected in a previous period. The knowledge of this quantity, in fact, is important for $\mathcal{C}_{\mathcal{A}}$ in the construction of suitable financial strategies.

In order to describe our model, we consider the spatial point process $R = \{(T_i, C_i)\}_{i\in\mathbb{N}}$, where the coordinates T_i and C_i represent, respectively, the arrival time and the size of the i-th claim. As natural, we assume that the random variables C_i's are i.i.d. and independent of the one-dimensional process $\{T_i\}_{i\in\mathbf{N}}$.

At time T_i, $\mathcal{C}_{\mathcal{A}}$ notifies $\mathcal{C}_{\mathcal{B}}$ on the received claim. After a random delay, $\mathcal{C}_{\mathcal{B}}$ will turn to $\mathcal{C}_{\mathcal{A}}$ a fixed percentage of the claim's size C_i, in agreement with the proportional part of the reinsurance contract.

The delay is considered to be random in that it depends on several factors, not completely under the control of the two companies. In particular, we assume here that such a delay is correlated with the claim's size C_i.

We thus obtain a transformed point process N, whose points represent the shares of single claims corresponded by $\mathcal{C}_{\mathcal{B}}$ to $\mathcal{C}_{\mathcal{A}}$ and the related delayed time.

The usual insurance models concern with a claims' arrival process that follows a Mixed Poisson Process. In our framework, the circumstance that the delay and the claim's size are correlated motivates our choice to treat R as a spatial point process. Some more explanation on this point will be given in the final section.

More specifically, we will assume R to be a SMPP and fix a ("baseline") *intensity measure* and a probability distribution for the *intensity-parameter*.

This assumption guarantees mathematical results that fit the financial intuition. Furthermore, the usual model where the arrival process is Mixed Poisson and the claims' sizes are i.i.d. can be obtained as a special case of our framework.

For our purposes, we need some results concerning random transformations of SMPP's. More precisely, we use the fact that a spatial point process obtained from a SMPP by means of a special type of transformation is still a SMPP (see Foschi and Spizzichino (2008)). In view of this result, we can easily deal with a model where the reinsurance contract is simultaneously of a proportional and of an excess-of-loss type by using these techniques.

The remaining part of the paper is organized as follows. In section 2 we introduce the basic notations and explain the mathematical setting. In Section 3 we present and discuss the insurance model built in the new framework of spatial point processes theory. In Section 4 we compute the conditional expected value of the amount to be received by $\mathcal{C}_{\mathcal{A}}$ from $\mathcal{C}_{\mathcal{B}}$ in a fixed time interval. This procedure is based on the computation of the expected value, of the number of points of the process N in a fixed region. Section 5 contains some conclusions. The Appendix is devoted to recalling some useful results on SMPP's.

2 Invariance property and parameter estimate for SMPP's

Let $R \equiv \{X_\alpha\}_{\alpha \in A}$, $X_\alpha \in \mathcal{X} \subseteq \mathbb{R}^k$ for any $\alpha \in A$, be a spatial point process (see e.g. Daley and Vere-Jones (1988), Stoyan, Kendall and Mecke (1995)) defined on some probability space $(\Omega, \mathcal{F}, P)$. With the symbol $R(I)$ we denote the cardinality of the set of points X_α such that $X_\alpha \in I$, with $I \in \mathbb{R}^k$ and $\alpha \in A$. We can suppose $A \subseteq \mathbb{N}$ (see Remark 10 in the Appendix). For any $\alpha \in A$, let W_α be a random variable defined on $(\Omega, \mathcal{F}, P)$ and taking values on a set $\mathcal{W} \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$. Furthermore, a transformation ϕ,

$$\phi : \mathcal{X} \times \mathcal{W} \to \mathcal{Y} \subseteq \mathbb{R}^k,$$

is given, such that $\phi(\cdot, w) : \mathcal{X} \to \mathcal{Y}$ is measurable and one-to-one for any fixed $w \in \mathcal{W}$. Now, we consider the transformed spatial point process $N \equiv \{Y_\alpha\}_{\alpha \in A}$ where

$$Y_\alpha = \phi(X_\alpha, W_\alpha). \tag{1}$$

We can also write

$$N = \Phi_\phi(R, \mathbf{W}), \tag{2}$$

where $\mathbf{W} = \{W_\alpha\}_{\alpha \in A}$.

We here concentrate our attention on the special case of Spatial Mixed Poisson Processes. These processes can be of wide use in applications because of their good mathematical properties and, at the same time, because of their flexibility, due to the randomness of the Poisson intensity.

Let $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), M)$ be a measure space, where $\mathcal{B}(\mathbb{R}^k)$ is the Borel σ-algebra and M is absolutely continuous with respect to the Lebesgue measure. We also introduce a probability distribution $U : [0, +\infty) \to [0, 1]$ of a r.v. Λ.

Definition 1 (SMPP)**.** *A spatial process D is Mixed Poisson with* mixing distribution U *and* baseline intensity measure *$M(\cdot)$ if, for $I \in \mathcal{B}(\mathbb{R}^k)$ and for $n \in \mathbb{N}$,*

$$P(D(I) = n) = \int_0^\infty e^{-\lambda M(I)} \frac{[\lambda M(I)]^n}{n!} dU(\lambda). \tag{3}$$

As mentioned in the Introduction, in this paper we assume that our process R is a SMPP. A main consequence of this assumption is that also its transformed process N in (2) belongs to the same class of the SMPP processes. The exact statement of this result, that was proved in Foschi and Spizzichino (2008), is reported as Theorem 12 in the Appendix. The main application of this fact concerns the following problem:

> let us fix a region $H \subseteq \mathcal{Y}$ and other two regions $I \subseteq \mathcal{X}$ and $J \subseteq \mathcal{Y}$. We need an estimate of $N(H)$, i.e. the number of points of N fallen in a region H, knowing the behavior of the processes R and N in the regions I and J respectively.

The rest of this section will be devoted to the solution of this problem.

For our problem, we would need a formula for the conditional probability of the event $\{N(H) = n\}$, given events of the type $\{R(I) = n', N(J) = n''\}$. Actually, this computation is complicated, because the sole information on $R(I)$ and $N(J)$ does not prevent us from counting points more than once. In fact, in order to evaluate the probability of the event $\{N(H) = n\}$, we also need to know the number $N(H \cap J)$ and the number of points of R fallen in I and sent by the transformation ϕ into J or into H. The latter random quantities will be denoted by $N_{(I)}(J)$ and $N_{(I)}(H)$ respectively, i.e., for $I \subseteq \mathcal{X}$ and $K \subseteq \mathcal{Y}$, we let

$$N_{(I)}(K) \equiv \sum_{\alpha \in A} \mathbf{1}_{\{\phi(X_\alpha, W_\alpha) \in K\}} \mathbf{1}_{\{X_\alpha \in I\}}, \tag{4}$$

(see also Remark 13 in the Appendix).

Concerning the regions H and J, it is shown in Foschi and Spizzichino (2008) that, without loss of generality, we can assume $H \cap J = \emptyset$ as a consequence of the *Order Statistic Property* of SMPP.

For our purposes, we need to know, instead, the number of the points X_α's such that $X_\alpha \in I$ and $\phi(X_\alpha, W_\alpha) \in J \cup H$; we then assume, more precisely, that we also observe an event of the type

$$E_{(l,m)} \equiv \{N_{(I)}(H) = l, N_{(I)}(J) = m\}, \quad l, m \in \mathbb{N} \cup \{0\}.$$

In this respect, we can state the following theorem:

Theorem 2. *For arbitrary subsets* $I,\ J,\ H$, *we have*

$$P(N(H) = n | R(I) = n', N(J) = n'', E_{(l,m)}) \tag{5}$$

$$= \int_0^\infty \frac{[\lambda M^*_{(\bar{I})}(H)]^{n-l}}{(n-l)!} e^{-\lambda M^*_{(\bar{I})}(H)} u(\lambda; I, J, n', n'', m) d\lambda,$$

where

$$u(\lambda; I, J, n', n'', m) = \frac{\lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J) + M(I)]} u(\lambda)}{\int_0^\infty \lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J) + M(I)]} u(\lambda) d\lambda} \tag{6}$$

and, for any $K \subseteq \mathcal{Y}$,

$$M^*_{(I)}(K) = \int_{\mathbb{R}^n} M(I \cap \phi_w^{-1}(K)) dG(w). \tag{7}$$

Proof. First of all, we assume, without loss of generality, that $H \cap J = \emptyset$.
We compute conditional probabilities of the type in (5) under the special condition $l = 0,\ m = 0$. Under this condition, we can write

$$P(N(H) = n | R(I) = n', N(J) = n'', E_{(l,m)}) =$$

$$= \frac{P(N(H) = n, N(J) = n'' | R(I) = n', E_{(l,m)})}{P(N(J) = n'' | R(I) = n', E_{(l,m)})} =$$

$$= \frac{\int_0^\infty \frac{[\lambda M^*(H)]^n}{n!} e^{-\lambda M^*(H)} \lambda^{n''} e^{-\lambda M^*(J)} \lambda^{n'} e^{-\lambda M(I)} u(\lambda) d\lambda}{\int_0^\infty \lambda^{n''} e^{-\lambda M^*(J)} \lambda^{n'} e^{-\lambda M(I)} u(\lambda) d\lambda} =$$

$$= \int_0^\infty \frac{[\lambda M^*(H)]^n}{n!} e^{-\lambda M^*(H)} u(\lambda; I, J, n', n'', m) d\lambda.$$

Actually, we now show that the latter formula can, however, be used to deal with the general situation, where l and m can be strictly positive.

In order to apply the previous formula to this case, we have to write the event $\{N(H) = n | R(I) = n', N(J) = n'', E_{(l,m)}\}$ in terms of suitable r.v.'s, such that the points numbered by them are counted only once.

Denoting with $\bar{I}$ the complementary set of I, the conditional probability in (5) is equal to

$$P\left\{N_{(\bar{I})}(H) = n - l, N_{(I)}(H) = l | N_{(\bar{I})}(J) = n'' - m, N_{(I)}(\overline{H \cup J}) = n' - l - m, E_{(l,m)}\right\}. \quad (8)$$

Then we have

$$P\left\{N_{(\bar{I})}(H) = n - l | N_{(\bar{I})}(J) = n'' - m, E_{(l,m)}, N_{(I)}(\overline{H \cup J}) = n' - l - m\right\} =$$

$$\frac{P\left\{N_{(\bar{I})}(H) = n - l, N_{(\bar{I})}(J) = n'' - m | E_{(l,m)}, N_{(I)}(\overline{H \cup J}) = n' - l - m\right\}}{P\left\{N_{(\bar{I})}(J) = n'' - m | E_{(l,m)}, N_{(I)}(\overline{H \cup J}) = n' - l - m\right\}}. \quad (9)$$

It is easy to prove that, for $H \cap J = \emptyset$, the conditioning events $E_{(l,m)}$ and $\{N_{(I)}(H \cup J) = l + m\}$ are equivalent (see also Foschi and Spizzichino (2008), where the *Order Statistic Property* of SMPP is explored). Thus

$$\{E_{(l,m)}, N_{(I)}(\overline{H \cup J}) = n' - l - m\}$$

is equivalent to $\{N_{(I)}(\mathbb{R}^k) = n'\}$, i.e. $\{R(I) = n'\}$. This last argument allows us to compute

$$P(N(H) = n | R(I) = n', N(J) = n'', E_{(l,m)}) =$$

$$= \frac{\int_0^\infty \frac{\lambda^{n-l+n''-m+n'} [M^*_{(\bar{I})}(H)]^{n-l}}{(n-l)!} e^{-\lambda [M^*_{(\bar{I})}(H) + M^*_{(\bar{I})}(J) + M(I)]} u(\lambda) d\lambda}{\int_0^\infty \lambda^{n''-m+n'} e^{-\lambda [M^*_{(\bar{I})}(J) + M(I)]} u(\lambda) d\lambda}, \quad (10)$$

that is the thesis. □

Remark 3. *Theorem 2 also provides an estimate of the parameter* Λ. *We notice, in fact, that* $u(\lambda; I, J, n', n'', m)$ *coincides with the posterior distribution of* Λ *given the observation of the event* $\{R(I) = n', N(J) = n'', N_{(I)}(J) = m\}$, *i.e.* $u(\lambda; I, J, n', n'', m) = u(\lambda | R(I) = n', N(J) = n'', N_{(I)}(J) = m)$.

3 The model

This section describes the aforementioned reinsurance model in a SMPP framework. We assume that the reinsurance contract between the companies $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{B}}$ becomes operative at time 0. Our analysis will be here restricted to a time interval

$$\mathcal{I} := [T^*, T^* + s]$$

with $T^* > 0$, i.e. we only consider the claims such that $T_i \in \mathcal{I}$. The company $\mathcal{C}_{\mathcal{A}}$ receives a claim of size C_i at a random time T_i according to a Mixed Poisson process, and T_i takes values in $\mathcal{I}$.

In view of the excess of loss part of the reinsurance contract, we assume that there exists a lower bound $\gamma > 0$ for the claims reinsured by $\mathcal{C}_{\mathcal{B}}$. Substantially, the claims must have a size large enough, in order to let $\mathcal{C}_{\mathcal{A}}$ exercise the reinsurance contract, and the threshold size is a deterministic constant, fixed by the insurance companies.

We then consider hereafter only claims of size larger than γ. We also consider an upper threshold Γ for the claim amount.

At time T_i, $\mathcal{C}_{\mathcal{A}}$ notifies $\mathcal{C}_{\mathcal{B}}$ on the received claim. After a random delay τ_i, $\mathcal{C}_{\mathcal{B}}$ will turn to $\mathcal{C}_{\mathcal{A}}$ a fixed percentage $v \in (0, 1)$ of the claim's size C_i, according to the proportional part of the reinsurance contract. In view of this fixed value v, the amount of repayment is contained in the interval $[v\gamma, v\Gamma]$. We then denote by $\Psi_i = v \cdot C_i$ the reimbursement received from $\mathcal{C}_{\mathcal{B}}$ for the i-th claim.

The times of repayments to $\mathcal{C}_{\mathcal{A}}$ are then given by $L_i \equiv T_i + \tau_i$. The delay τ_i is random in that it depends on several factors, not completely under the control of the two companies. In particular, we assume here that τ_i is related to the claim's size C_i. More precisely, for fixed $i \in \mathbb{N}$, we consider the delay as a positive r.v.'s, defined by $\tau_i := C_i W_i$, where $\{W_i\}_{i\in\mathbb{N}}$ are positive r.v., i.i.d. and independent of the process $\{(T_i, C_i)\}_{i\in\mathbb{N}}$.

We now consider the spatial point process

$$R = \{(T_i, C_i)\}_{i\in\mathbb{N}}.$$

Furthermore, for given $v \in (0, 1)$, we denote by ϕ_v the transformation

$$\phi_v : \mathbb{R}_+ \times [\gamma, \Gamma] \times \mathbb{R}_+ \to \mathbb{R}_+ \times [v\gamma, v\Gamma]$$

defined by

$$\phi_v((T_i, C_i), W_i) = (T_i + C_i W_i, vC_i) = (L_i, \Psi_i). \tag{11}$$

By means of the transformation ϕ_v, we then define the process

$$N = \{(L_i, \Psi_i)\}_{i\in\mathbb{N}}.$$

Remark 4. *We claim that N is still a Mixed Poisson process. This can be obtained as a consequence of the measurability of the transformation ϕ_v in (11) and of Theorem 12 in the Appendix.*

We also point out that, being a SMPP, N is a simple process (see Lemma 9 and Remark 10). This property fits well with the model we described, since it implies that an infinite number of claims cannot occur in a bounded region.

The transformation ϕ_v in (11) describes the dependence between the pairs (T_i, C_i) and (L_i, Ψ_i). A different assumption on the structure of dependence between the involved processes, and then a consequent different definition of ϕ_v, has an influence on the distributional result for N. However, N remains a SMPP as long as ϕ_v is measurable. Therefore, we can argue that our model allows us to consider a wide range of types of stochastic dependence among claims' sizes and reimbursements' delays. Nevertheless, the particular choice of ϕ_v in (11) seems to be in agreement with the empirical evidence and the existing literature.

Remark 5. *In our model we have $\tau_i = \psi(C_i, W_i) = C_i W_i$, so that ψ is strictly increasing in both the variables.*
It turns out that τ_i is stochastically increasing (SI) in C_i (see e.g. Joe, 1997). This means

$$P(\tau_i \geq s | C_i = c) \uparrow c.$$

$$P(\tau_i \geq s | C_i = c) = P(W_i \geq \frac{s}{C_i} | C_i = c)$$

and, by the assumption of independence between W_i and C_i, we can write

$$P(W_i \geq \frac{s}{C_i} | C_i = c) = P(W_i \geq \frac{s}{c}) = 1 - G(\frac{s}{c}), \tag{12}$$

where G is the distribution function of W_i. At this point, since $1 - G(\frac{s}{c})$ is increasing in c, we can conclude by (12) that τ_i is SI in C_i independently of the probability distributions of C_i and W_i. The distribution of W_i can only affect the strength of the increasingness.

More generally, for any ψ, strictly increasing (or decreasing) in both the variables, $\tau_i = \psi(C_i, W_i)$ is SI in C_i.

The fact that τ_i is SI with respect to C_i implies that the delay τ_i grows with the claim's size.

4 An estimation result

Let us consider the regions

$$I \equiv \mathcal{I} \times [\gamma, \Gamma], \qquad J \equiv \mathcal{I} \times [v\gamma, v\Gamma]$$

and define the random subset of indexes $\{i_1, \ldots, i_K\} \subset \mathbb{N}$, such that

$$\{(T_{i_1}, C_{i_1}), \ldots, (T_{i_K}, C_{i_K})\} = I \cap R.$$

We notice that $K \equiv |I \cap R| = R(I)$. We also consider the random subset of indexes $\{i'_1, \ldots, i'_{K'}\} \subseteq \{i_1, \ldots, i_K\}$ such that

$$\{\phi_v((T_{i'_1}, C_{i'_1}), W_{i'_1}), \ldots, \phi_v((T_{i'_{K'}}, C_{i'_{K'}}), W_{i'_{K'}})\} = J \cap N.$$

Similarly to above, we notice that $K' \equiv |J \cap N| = N(J)$. Since the two regions $I,\ J$ have the same projection $\mathcal{I}$ on the time axis, we can conclude that

$$P(K' \leq K) = 1.$$

We consider now a time interval $\mathcal{H} = [\widetilde{T}, \widetilde{T} + r]$, with $r > 0$ and $\widetilde{T} = T^* + s$, and we define the rectangle

$$H \equiv \mathcal{H} \times [v\gamma, v\Gamma].$$

The points of N belonging to H represent the claims reimbursed by $\mathcal{C_B}$ to $\mathcal{C_A}$ in the time interval $\mathcal{H}$, in force of the reinsurance contract. Actually, we are interested in the estimation of the amount that the company $\mathcal{C_B}$ will turn to $\mathcal{C_A}$ in the time interval $\mathcal{H}$. Let $\Upsilon = \{j_1, \ldots, j_{K''}\}$ be the random subset of indexes such that

$$\{(L_{j_1}, \Psi_{j_1}), \ldots, (L_{j_{K''}}, \Psi_{j_{K''}})\} = H \cap N,$$

with $K'' \equiv |H \cap N| = N(H)$.

Remark 6. *Since H is a normal domain, the condition $(L_j, \Psi_j) \in H$ is equivalent to*

$$\begin{cases} L_j \in \mathcal{H}, \\ \Psi_j \in [v\gamma, v\Gamma]. \end{cases}$$

We then denote by Q the amount of the aggregate claim that $\mathcal{C_B}$ corresponds to $\mathcal{C_A}$ during $\mathcal{H}$, i.e.

$$Q \equiv \sum_{j \in \Upsilon} \Psi_j.$$

In this section, we will be dealing with the estimation of Q on the basis of the information collected in the period $\mathcal{I}$. More precisely, we will approximate the conditional expectation of Q given

- the number $R(I)$ of the claims requested to the company $\mathcal{C_A}$ during the interval $\mathcal{I}$ and whose size is larger than γ;
- the number $N(J)$ of reimbursements made by $\mathcal{C_B}$ to $\mathcal{C_A}$ during $\mathcal{I}$;
- the number of claims received by $\mathcal{C_A}$ during $\mathcal{I}$ and reimbursed by $\mathcal{C_B}$ to $\mathcal{C_A}$ during $\mathcal{I}$. According to formula (4), this quantity is $N_{(I)}(J)$ and, obviously, $N_{(I)}(J) \leq N(J)$.

Remark 7. *We notice that the sets $\{i'_1, \ldots, i'_{K'}\},\ \{i_1, \ldots, i_K\}$ are countable and, moreover, finite with probability 1. Hence, we also have $\mathbb{E}[R(I)] < +\infty$ and $\mathbb{E}[N(J)] < +\infty$. This follows by Lemma 9 and by the fact that the regions $I,\ J$ are bounded.*

As we shall see, the approximation of the expected value of Q can be obtained by applying Theorem 2.

We proceed by considering the following partition of H:

$$\Delta_k := \{H_s^{(k)}\}_{s=1,\ldots,k}, \qquad k \in \mathbb{N},$$

where

$$H_s^{(k)} := \mathcal{H} \times (c_{s-1}^{(k)}, c_s^{(k)}],$$

with

$$c_s^{(k)} = v\gamma + \frac{s}{k}(v\Gamma - v\gamma), \qquad s = 1, \ldots, k.$$

The smaller is the length of the interval $(c_{s-1}^{(k)}, c_s^{(k)}]$, the better is the approximation of Ψ_j.

We denote by $a_s^{(k)}$ the expected number of claims that will be paid by $\mathcal{C_B}$ to $\mathcal{C_A}$ in $\mathcal{H}$ with reimbursement amounts in $(c_{s-1}^{(k)}, c_s^{(k)}]$, for each $s = 1, \ldots, k$, conditional on the information collected in the previous period $\mathcal{I}$, i.e.

$$a_s^{(k)} \equiv \mathbb{E}\Big[N(H_s^{(k)})|R(I) = n', N(J) = n^{''}, N_{(I)}(J) = m\Big]. \tag{13}$$

The next result provides a closed form expression to compute $a_s^{(k)}$, for any $k \in \mathbb{N}$ and $s = 1, \ldots, k$.

Proposition 8.

$$a_s^{(k)} = \sum_{n=0}^{+\infty}\sum_{l=0}^{n} n \frac{[M^*_{(\bar{I})}(H_s^{(k)})]^{n-l}}{(n-l)!} \frac{[M^*_{(I)}(H_s^{(k)})]^{l}}{l!} \int_0^{+\infty} \lambda^{n-l} e^{-\lambda M^*_{(\bar{I})}(H_s^{(k)})} u(\lambda; I, J, n', n'', m) \mathrm{d}\lambda \cdot$$
$$\cdot \int_0^{+\infty} \lambda^{l} e^{-\lambda M^*_{(I)}(H_s^{(k)})} u(\lambda; I, J, n', n'', m) \mathrm{d}\lambda,$$

where

$$u(\lambda; I, J, n', n'', m) = \frac{\lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J)+M(I)]} u(\lambda)}{\displaystyle\int_0^{\infty} \lambda^{n''-m+n'} e^{-\lambda[M^*_{(\bar{I})}(J)+M(I)]} u(\lambda) d\lambda}$$

is the posterior distribution on Λ defined in Eq. 6.

Proof. The proof is articulated in three steps.

First of all, in Eq. 5, we replace the subset H with $H_s^{(k)}$. Then we write

$$P(N(H_S^{(K)}) = n|R(I) = n', N(J) = n'', N_{(I)}(J) = m)$$
$$= \sum_{l=0}^{+\infty} P(N(H_S^{(K)}) = n|R(I) = n', N(J) = n'', N_{(I)}(H_S^{(K)}) = l, N_{(I)}(J) = m) \cdot$$
$$\cdot P(N_{(I)}(H_S^{(K)}) = l|R(I) = n', N(J) = n'', N_{(I)}(J) = m),$$

where we remove the conditioning on the event $N_{(I)}(H_S^{(K)}) = l$ by summing on the index l. We obtain

$$P(N(H_S^{(K)}) = n|R(I) = n', N(J) = n'', N_{(I)}(J) = m) =$$

$$\sum_{l=0}^{n} \frac{[M^*_{(\bar{I})}(H_s^{(k)})]^{n-l}}{(n-l)!} \cdot \int_0^{+\infty} \lambda^{n-l} e^{-\lambda M^*_{(\bar{I})}(H_s^{(k)})} u(\lambda; I, J, n', n'', m) \mathrm{d}\lambda \cdot$$

$$\cdot \frac{[M^*_{(I)}(H_s^{(k)})]^{l}}{l!} \int_0^{+\infty} \lambda^{l} \cdot e^{-\lambda M^*_{(I)}(H_s^{(k)})} u(\lambda; I, J, n', n'', m) \mathrm{d}\lambda. \tag{14}$$

The last step consists in computing the expected value in Eq. (13) by taking into account (14). □

By using Proposition 8, we can now provide an upper and a lower approximation of $\mathbb{E}[Q|R(I) = n', N(J) = n'', N_{(I)}(J) = m]$.
In fact, by letting

$$\bar{\theta}_k := \sum_{s=1}^{k} c_s^{(k)} a_s^{(k)},$$

$$\underline{\theta}_k := \sum_{s=1}^{k} c_{s-1}^{(k)} a_s^{(k)}.$$

We have, for any $k \in \mathbb{N}$,

$$\underline{\theta}_k \le \mathbb{E}[Q|R(I) = n', N(J) = n'', N_{(I)}(J) = m] \le \bar{\theta}_k. \tag{15}$$

$\{\underline{\theta}_k\}$ is non-decreasing and $\{\bar{\theta}_k\}$ is non-increasing with respect to k. Moreover, there exists a constant $q > 0$ such that

$$\lim_{k\to+\infty} \underline{\theta}_k = \lim_{k\to+\infty} \bar{\theta}_k = q.$$

By (15) we can then conclude that

$$\mathbb{E}[Q|R(I) = n', N(J) = n'', N_{(I)}(J) = m] = q.$$

5 Concluding remarks

This paper deals with a reinsurance model in a SMPP framework. We assume that the reinsurance contract between two companies is a combination of contracts of two types: excess of loss and proportional. After analyzing some aspects of the theory of the SMPP's, we obtain an estimation result for the amount that the ceding company receives from the reinsurer, within a specified time interval. More precisely, we provide an estimate of the aggregate claim in a period, given the observation of claims and payments in a previous period.

Our approach is based on the invariance of stochastic structure of the process N with respect to R. The lack of independence between delays and claims' sizes imposes us to abandon the usual Mixed Poisson framework and to treat the claims' arrival process as a spatial process.
In fact, consider the familiar simple model described as follows:

(i) the claims' arrival times are described by a one-dimensional Mixed Poisson Process;

(ii) the claims' sizes are i.i.d. and independent of the arrival times;

(*iii*) the delays are i.i.d. and independent from the process R.

Then, the process N has the same stochastic structure described by (*i*) and (*ii*).

Generally this is not true anymore in the case when condition (*iii*) fails. As argued, in our treatment it is instead the stochastic structure of SMPP that remains invariant, even in the case of stochastic dependence between delays and the process R.

We point out that, from a mathematical point of view, our discussion could be extended as follows:

- the regions H are of more general shape than rectangles;
- time dependence between the upper and lower thresholds of the claims, γ and Γ, can be introduced. This could be the starting point for an extension of Theorem 12.

Our present assumptions, however, are convenient, in order to treat reinsurance models in agreement with empirical evidence and existing literature.

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Appendix

Among several properties of SMPP's, we recall here two important ones, that have been used or can be useful in better understanding the subject of the paper and some implications of theirs. These results are well known in the case of Poisson processes on the line (see Renyi (1967)) and can be easily extended to spatial Poisson processes and, subsequently, to spatial Mixed Poisson processes.

Lemma 9. *A Spatial Mixed Poisson process D is a simple point process.*

Proof. Since $M(\cdot)$ is absolutely continuous with respect to Lebesgue measure, if I has null Lebesgue measure, then

$$P(D(I) = n) = \begin{cases} 0 & n = 1, 2, \dots, \\ 1 & n = 0. \end{cases}$$

□

Remark 10. *By the simplicity of the process also the countability of its points follows. Hence we can assume the index set A to be a subset of $\mathbb{N}$.*

By this, it follows that, for any bounded set I, $D(I)$ is finite with probability 1. Another consequence of Lemma 9 is that $\mathbb{E}[D(I)] < +\infty$, for any bounded set I.

Another important property, stated in the following theorem, is the invariance of SMPP's under suitable random transformations. It is already known in literature (see e.g. Cinlar (1995)), but we state it here in a form that is convenient for our use.
A proof, grounded on geometrical arguments and based on the Order Statistic Property of SMPP's, has been provided in Foschi and Spizzichino (2008). It is based on the following property, known in the one-dimensional case as conditional independence of increments.

Lemma 11. *Let $I_1, \ldots, I_m$ be disjoint subsets of $\mathbb{R}^k$, with $I_1, \ldots, I_m \in \mathcal{B}$; N is a spatial Mixed Poisson process if and only if $N(I_1), \ldots, N(I_m)$ are conditionally independent and Poisson distributed given Λ.*

Theorem 12. *Let R be a SMPP with mixing distribution U and baseline intensity measure M. Let furthermore be $\mathbf{W} = \{W_\alpha\}$ be i.i.d., with distribution G and independent of R. Then $N = \Phi_\phi(R, \mathbf{W})$ is a SMPP with the same mixing distribution U and intensity measure*

$$M^*(J) = \int_{\mathbb{R}^n} M(\phi_w^{-1}(J))dG(w).$$

Remark 13. *For a given $I \subseteq \mathcal{X}$, $N_{(I)}$ can be thought of as a SMPP of its own, with mixing distribution U and baseline intensity measure*

$$M^*_{(I)}(J) = \int_{\mathbb{R}^n} M(I \cap \phi_w^{-1}(J))dG(w).$$